المستخلص

في هذه الأطروحــة، درسنا تأثير المجـال المغناطيســي الهيدروديناميكــي علـى التدفقات المتسارعة للموائع اللزجة مع نموذج "بيركر". والذي يصف حقل سرعة التدفق بواسطة معادلة تفاضليه جزئية كسرية. استخدمنا تحويـلات كل مـن فوريـر و لابــلاس، للحصول على الحلول الدقيقة لتوزيع السرعة للمسألتين التاليتين : التدفق الناجم عن لوحة التسريع الثابتة، و التدفق الناجم عن لوحة التسريع المتغيرة. هذه الحـلول، كتبت بصيغـة التكامل والمتسلسلات بدلالة الدالة ميتاج لفلر،كما انها تظهر بصيغة جمع للجزئين. الجزء الاول يمثل حقل السرعة للموائع النيوتونية لأداء الحركة نفسها، والجزء الثاني يمثل الاضافة الى حقل سرعة المائع النيوتوني وهذا لسبب كونه المائع الذي درسناه هو مائع لانيوتوني. تم الحصول على حلول مماثلة لموائع من الرتبة الثانية مثل ماكسويل، و اولدرويد من النمط بي.ذات مشتقات كسرية، بالأضافة الى ذلك، وكحالات خاصة، تم تغطيتها، هي عندما

$$\alpha = \beta = 1$$

كمــا كان متوقعا، حلولنا تميل الى حلـول مماثلة لموائع بيركــر الأولية. بينما برنامج الماثيماتيكا أستعمل لرسم اشكال مكونات السرعة في المستوي.

جمهورية العراق
وزارة التعليم العالي
والبحث العلمي
جامعة بغداد
كلية العلوم

تأثيــر المجال المغناطيسي الهيدروديناميكــي على التدفقات المتسارعة للموائع اللزجة المرنـة مـــع نمـوذج بيركـــر للمشتقات الكسرية

رسالة مقدمة

الى

جامعة بغداد/ كلية العلوم

كجزء من متطلبات نيل شهادة الماجستير في علوم الرياضيات

من قبل

هند شاكــر محمـــود

٢٠١٢

Republic of Iraq
Ministry of Higher Education
And Scientific Research
University of Baghdad
College of Science

EFFECT OF MHD ON ACCELERATED FLOWS OF A VISCOELASTIC FLUID WITH THE FRACTIONAL BURGERS' MODEL

A Thesis
Submitted to the College of Science
University of Baghdad
in Partial Fulfillment of The Requirements
for The Degree of Master of Science in
Mathematics

By

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Supervisor

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2012

Acknowledgements

Best Praise to Allah, most merciful for helping me and giving he patience along the life.

I take this opportunity as my privilege to express my sincere gratitude towards my thesis supervisor Dr. Ahmed MawloodAbdulhadi, for his advice, guidance, and invaluable suggestions throughout the entire period of my work. He has been a constant source of inspiration and encouragement.

I also wish to express my thanks to all the staff of Department of Mathematics.

Last but not least, I express my deep gratitude to my family and my heartfelt thanks to my friends for their continuous interest to complete this work.

Hind ShakirMahmood

2012

الإهداء

الإهداء

إلى من يسعد قلبي بلقياها
إلى روضة الحب التي تنبت أزكى الأزهار

أمي

إلى رمز الرجولة والتضحية

إلى من دفعني إلى العلم وبه ازداد افتخار

أبي

إلى من هم اقرب إليّ من روحي
إلى من شاركني حضن ألام وبهم استمد عزتي وإصراري

إخوتي

إلى من آنسني في دراستي وشاركني همومي
تذكاراً وتقديراً

أصدقائي

أهدي ثمرة جهدي

هند شاكر محمود

٢٠١٢

"بسم الله الرحمن الرحيم"

((وَقُلِ اعْمَلُوا فَسَيَرَى اللَّهُ عَمَلَكُمْ وَرَسُولُهُ وَالْمُؤْمِنُونَ))

((صدق الله العظيم))

الآية ١٠٥ من سورة التوبة

Certification

I certify that this thesis "**EFFECT OF MHD ON ACCELERATED FLOWS OF A VISCOELASTIC FLUID WITH THE FRACTIONAL BURGERS' MODEL**"was prepared under my supervision at University of Baghdad, College of Science, Department of Mathematics as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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In view of the available recommendations, I forward this thesis for debate by the examining committee.

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Title: Assistant Professor, H. O. D

Date: /   /

Committee Certification

We certify that we have read this thesis "**EFFECT OF MHD ON ACCELERATED FLOWS OF A VISCOELASTIC FLUID WITH THE FRACTIONAL BURGERS' MODEL** " and, as an examining committee examined the student in its context and that in our opinion it is adequate for the partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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Contents

Abstract

In this thesis, we studied the effect of MHD on accelerated flows of a viscoelastic fluid with the fractional Burgers' model. The velocity field of the flow is described by a fractional partial differential equation. By using Fourier sine transform and Laplace transform, an exact solutions for the velocity distribution are obtained for the following two problems: flow induced by constantly accelerating plate, and flow induced by variable accelerated plate. These solutions, presented under integral and series forms in terms of the generalized Mittag-Leffler function, are presented as the sum of two terms. The first terms represent the velocity field corresponding to a Newtonian fluid, and the second terms give the non-Newtonian contributions to the general solutions. The similar solutions for second grad, Maxwell and Oldroyd-B fluids with fractional derivatives as well as those for the ordinary models are obtained as the limiting cases of our solutions. Moreover, in the special cases when $\alpha = \beta = 1$, as it was to be expected, our solutions tend to the similar solutions for an ordinary Burgers' fluid. While the MATHEMATICA package is used to draw the figures velocity components in the plane.

Introduction

A fluid is that state of matter which capable of changing shape and is capable of flowing. Both gases and liquids are classified as fluid, each fluid characterized by an equation that relates stress to rate of strain, known as “State Equation”. And the number of fluids engineering applications is enormous: breathing, blood flow, swimming, pumps, fans, turbines, airplanes, ships, pipes… etc. When you think about it, almost every thing on this planet rather is a fluid or moves with respect to a fluid.

Fluid mechanics is considered a branch of applied mathematics which deal with behavior of fluids either in motion (fluid dynamics) or at rest (fluid statics).

Within the past fifty years, many problems dealing with the flow of Newtonian and non-Newtonian fluids through porous channels have been studied by engineers and mathematicians. The analysis of such flows finds important applications in engineering practice, particularly in chemical industries, investigations of such fluids are desirable. A number of industrially important fluids including molten plastics, polymers, pulps, foods and fossil fuels, which may saturate in underground beds, display non-Newtonian behavior. Examples, of such fluids, second grade fluid is the simplest subclass for which one can hope to gain an analytic solution.The MHD phenomenon is characterized by an interaction between the hydrodynamic and boundary layer electromagnetic field.

The study of MHD flow in a channel also has application in many devices like MHD power generators, MHD pumps, accelerators, etc. As to the history of fractional calculus, already in 1965 L'Hospital raised the question as to meaning of $d^n y/dx^n = 1/2$, that is "what if n is fractional?". "This is an apparent paradox from which, one day, useful consequences will be drawn", Leibniz replied, together with "$d^{1/2}x$ will be equal to $x\sqrt{dx:x}$". S. F. Lacroix was the first to mention in some two pages a derivative of arbitrary order in a 700 page text book of 1819.

Thus for $y = x^a, a \in \Re_+$, he showed that

$$\frac{d^{1/2}y}{dx^{1/2}} = \frac{\Gamma(a+1)}{\Gamma(a+1/2)} x^{a-1/2} .$$

In particular he had $(d/dx)^{1/2} x = 2\sqrt{x/\pi}$ (the same result as by the present day Riemann-Liouville definition below). J. B. J. Fourier, who in 1822 derived an integral representation for $f(x)$,

$$f(x) = \frac{1}{2\pi}\int_{\Re} f(\alpha)d\alpha \int_{\Re} \cos p(x-\alpha)dp,$$

obtained (formally) the derivative version

$$\frac{d^v}{dx^v} f(x) = \frac{1}{2\pi}\int_{\Re} f(\alpha)d\alpha \int_{\Re} p^v \cos\{p(x-\alpha)+\frac{v\pi}{2}\}dp,$$

Where "the number v will be regarded as any quantity whatever, positive or negative".

It is usually claimed that Abel resolved in 1823 the integral equation arising from the brachistochrone problem, namely

$$\frac{1}{\Gamma(\alpha)}\int_0^x \frac{g(u)}{(x-u)^{1-\alpha}}du = f(x), \quad 0<\alpha<1$$

With the solution

$$g(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x \frac{f(u)}{(x-u)^{\alpha}}du$$

As J. Lutzen first showed, Abel never solved the problem by fractional calculus but merely showed how the solution, found by other means, could be written as a fractional derivative. Lutzen also briefly summarized what Abel actually did. Liouville, however, did solve the integral equation in 1832. Fractional calculus has developed especially intensively since 1974 when the first international conference in the field took place. It was organized by Betram Ross and took place at the university of New Haven, Connecticut in 1974. It had an exceptional turnout of 94 mathematicians; the proceedings contain 26 papers by the experts of the time. It was followed by the conferences conducted by Adam Mc Bride and Garry Roach (University of Strathclyde, Glasgow, Scotland) of 1989, by Katsuyuki Nishimoto (Nihon University, Tokyo, Japan) of 1989, and by Peter Rusev, Ivan Dimovski and Virginia Kiryakova (Varna, Bulgaria) of 1996. In the period 1975 to the present, about 600 papers have been published relating to fractional calculus [9].

Understanding non- Newtonian fluid flows behavior becomes increasingly important as the application of non-Newtonian fluids perpetuates through various industries, Including polymer processing and electronic packaging , paints , oils liquid polymers, glycerin , chemical , geophysics , biorheology. However, there is no model which can alone predict the behaviors of all non-Newtonian fluids. Amongst the existing model, rate type models have special importance and many researchers are using equations of motion of Maxwell and Oldroyd-fluid flows. M. Khan, S. Hyder Ali, Haitao Qi. (2007) [10] construct the exact solutions for the accelerated flows of a generalized Oldroyd-B fluid. The fractional calculus approach is used in the constitutive relationship of a viscoelastic fluid. The velocity field and the adequate tangential stress that is induced by the flow due to constantly accelerating plate and flow due to variable accelerating plate are determined by means of discrete Laplace transform. C. Fetecau,T. Hayat and M.Khan. (2008) [5] concerned with the study of unsteady flow of an Oldroyed-B fluid produced by a suddenly moved plane wall between two side walls perpendicular to the plane are established by means of the Fourier sine transforms. M. Khan, S. Huder Ali, Haitao Qi. (2009) [11] Studied the accelerated flows for a viscoelastic fluid governed by the fractional Burgers' model. The velocity field of the flow is described by a fractional partial differential equation. Liancun Zheng, Yaqing Liu, Xinxin Zhang. (2011) [13] research for the magnetohydrodynamic (MHD) flow of an incompressible generalized Oldroyd-B fluid due to an infinite accelerating plate. The motion of the fluid is produced by the infinite plate, which at time $t = 0^{+}$ begins to slide in its plane with a velocity At. The solutions are established by means of Fourier sine and Laplace transforms.

This thesis contains three chapters:-

In chapter one, we introduced an elementary concepts and basic definitions that we will use in our work.

Chapter two contains the statement of the problem of the flow induced by a constant accelerated plate, the solution of the problem, and results and discussion of the problem. Laplace transformation and Fourier transformation are used to solve the problem.

Chapter three contains the statement of the problem of the flow induced by a variable accelerated plate, the solution of the problem, and results and discussion of the problem. Laplace transformation and Fourier transformation are used to solve the problem.

Chapter One

Basic Definitions and Elementary Concepts

Introduction

In this chapter, we give some basic definitions and elementary concepts that we will be used in our work latter on.

(1.1)Fluid mechanics:[14]

The subject of fluid mechanics deals with the behavior of fluids when subjected to a system of forces. The subject can be divided in to three fields:

I-Statics: which deals with the fluid elements which are at rest relative to each other.

II-kinematics: This deals with the effect of motion. i.e. translation, rotation and deformation on the fluid elements.

III-Dynamics: This deals with the effect of applied forces on fluid elements.

Fluid (1.1.1):[19]

It is defined as a substance that continuous to deform when subjected to a shear stress, no matter how small.

Mass density (1.1.2) : [14, 20]

It is defined as mass per unit volume of fluid, and denoted by ρ. Mathematically

$$\rho = \frac{m}{v} \left(\frac{kg}{m^3}\right) \qquad (1-1)$$

Where, ρ = density

m = mass

v = volume

Pressure (1.1.3): [14, 20]

The pressure, denoted by p, is a normal compressive force per unit area.

$$p = \frac{Force}{Area} \left(\frac{kg}{m.s^2}\right) \qquad (1-2)$$

Where force equals mass times acceleration.

Shear stress (1.1.4):[15]

It is defined as the force per unit area, mathematically:

$$T = \frac{F}{A} \qquad (1-3)$$

Where, T = shear stress

F = the Force applied

A = the cross-sectional area of material with area parallel to the applied Force vector.

Shear strain (1.1.5): [15]

Also known as shear a deformation of solid body is displaced parallel planes in the body; quantitatively it is the displacement of any plane relative to a second plane divided by the perpendicular distance between planes the force causing such deformation.

Newton's law of viscosity(1.1.6): [1, 3]

The Newton's law of viscosity states that the shear stress (T) fluid element on a layer as directly is proportional to the shear strain or gradient:

$$T \propto \frac{du}{dy} \qquad (1-4)$$

This may be written as:

$$T = \mu \frac{du}{dy} \qquad (1-5)$$

Where μ a constant of proportionality is called "dynamic viscosity ".

The dimensions may be found as follows:

$$\mu = \frac{T}{du/dy} = \frac{stress}{velocity/Distance} = \frac{Force}{Area} \times \frac{Distance}{velocity} \qquad (1-6)$$

Viscosity (1.1.7): [14, 20, 2]

Viscosity is the resistance of a fluid to motion- it's internal friction.

A fluid in a static state is by definition unable to resist even the slightness amount of shear stress. Application of shear stress results in a continual and permanent distortion known as flow.

Dynamic viscosity (1.1.8): [2]

A dynamic viscosity μ is defined as the tangential force required per unit area to sustain a unit velocity gradient.

$$T = \mu \frac{du}{dy} \qquad (1-7)$$

Where τ is the sheer stress (force per unit area) $\frac{du}{dy}$ is called a velocity gradient and μ is the coefficient of dynamic viscosity, or simply called viscosity.

Kinematics viscosity (1.1.9): [2]

Is defined as the ratio of dynamic viscosity to mass density and denoted by υ . Mathematically

$$\upsilon = \frac{\mu}{\rho}\left(\frac{l^2}{T}\right) \qquad (1-8)$$

Where l standing for length.

Classification of fluids (1.1.10): [14]

The fluid may be classified into the following types depending upon the presence of viscosity.

Ideal fluid (1.1.10.1) (In viscid)

Such a fluid, will not offer any resistance to displacement of surface in contact (i.e. T = 0) where T is the shear stress.

Real fluid(1.1.10.2)

Such fluid will always resist displacement.

Newtonian fluid (1.1.10.3)

A real fluid in which shear stress is directly proportional to the rate of shear strain. i.e. (obeys the Newton's law of viscosity).

Non-Newtonian fluid (1.1.10.4)

A real fluid in which shear stress is not directly proportional to the rate of shear strain (non linear relation).i.e. dose not obey the Newton's law of viscosity.

(1.1.11)Reynolds number: [14, 19]

The Reynolds number, denoted by Re, is dimensionless and represents the ratio of inertia force to the viscous force and given by:

$$\mathrm{Re} = \frac{Vd\rho}{\mu} = \frac{Vd}{\upsilon} \qquad (1-9)$$

Where d is standing for distance. The use of Reynolds number is to determine the nature of flow whether is laminar (Re < 2000) or turbulent (Re > 4000).

Types of fluid flow(1.1.12): [14]

A fluid flow consists of flow of number of small particles grouped together. These particles may group themselves in variety of ways and type of flow depends on how these groups behave. The following are important types of fluid flow:

I- Steady and Unsteady Flow: a flow is considered to be steady when conditions at any point in the fluid flow do not change with time i.e.

$$\frac{\partial V}{\partial t} = 0,$$

and also the properties do not change with time; i.e.

$$\frac{\partial p}{\partial t} = 0, \frac{\partial \rho}{\partial t} = 0.$$

Otherwise the flow is unsteady.

II- Compressible and Incompressible Flow: a flow is considered to be compressible if the mass density of fluid ρ changes from point to point, or ρ ≠ constant. In case of incompressible flow the change of mass density in the fluid is neglected or density is assumed to be constant.

III- Laminar and Turbulent Flow: Laminar flow in which fluid particles move along smooth paths in laminar or layers, with one layer gliding smoothly over an adjacent layer and it occurs for values of Reynold's number from 0 to 2000. And we say that the flow is turbulent flow if the fluid particles move in very Irregular parts and when Reynold's number is greater than 4000, and we say that the flow is translation if the values of Reynolds number between 2000 and 4000.

Continuity equation(1.1.13): [21]

The continuity equation simply expresses the law of conservation of mass (the mass per unit time entering the tube must flow out at same rate) mathematical form:

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=o \qquad (1-10)$$

Where, ρ is density and (u, v, w) are the velocity components in (x, y, z) direction, respectively.

If the fluid is compressible, then ρ is constant, and the continuity equation may be written as:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (1-11)$$

In 2-dimension:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (1-12)$$

In 1-dimension:

$$\frac{\partial u}{\partial x}=0 \qquad (1-13)$$

Motion equations (1.1.14): [6]

It is a system of partial differential equations that describe the fluid motion. The general technique for obtaining the equations governing fluid motion is to consider a small control volume through which the fluid moves and require that mass and energy are conserved , and that the rate of change of the two components of linear momentum are equal to the corresponding components of the applied force.

The Navier-stokes equation(1.1.15):[21]

The system of partial differential equations that describe the fluid motion is called the Navier- Stokes equations. The general technique for obtaining the equations governing fluid motion is to consider a small control volume through which the fluid moves, and required that mass and energy are conserved, and that the rate of change of the two components of linear momentum are equal to the corresponding components of the applied force. The Navier-Stokes equations for incompressible fluid are:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=X-\frac{1}{p}\frac{\partial \rho}{\partial x}+v(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=Y-\frac{1}{p}\frac{\partial \rho}{\partial y}+v(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2})$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=Z-\frac{1}{p}\frac{\partial \rho}{\partial z}+v(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2})$$

Where (u, v, w) are the velocity components in the x, y and z directions respectively, (X, Y, Z) are the body force in the x, y and z directions respectively, ρ is the mass density, p is the pressure and v is the kinematic velocity.

(1.2)Laplace transform methods: [4]

The Laplace transformation is a powerful method for solving linear differential equations (partial or ordinary) arising in engineering mathematics. It is consists essentially of three steps. In the first step, the given partial differential equation is transformed into an ordinary differential equation (subsidiary equation). Then the resulting equation is solved by usual methods. Finally, the solution of the subsidiary equation is transformed back so that it becomes the required solution of the original differential equation. In the case of ordinary differential equation, the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem. Another advantage is that it takes care of initial conditions so that in initial value problems the determination of a general solution is avoided. Similarly, if we apply the Laplace transformation to a non-homogeneous equation, we obtain the solution directly, that is, without first solving the corresponding homogeneous equation.

Laplace and inverse transform (1.2.1):

Let $f(t)$ be a given function which is defined for all positive values of t. We multiply $f(t)$ by $\exp(-st)$ and integrate with respect to t from zero to infinity.

Then, if resulting integral exists, it is a function of s, say, $F(s)$:

$$F(s)=\int_0^\infty \exp(-st)\, f(t)\,dt.$$

The function $F(s)$ is called the Laplace transform of the original function $f(t)$, and will be denoted by $L(f)$. Thus

$$F(s)=L(f)=\int_0^\infty \exp(-st)\, f(t)\,dt \qquad (1-14)$$

The described operation on $f(t)$ is called the Laplace transformation. Furthermore the original function $f(t)$ in (1-14) is called the inverse transform or inverse of $F(s)$ and will be denoted by $L^{-1}(F)$; that is, we shall write

$$f(t)=L^{-1}(F)\,.$$

<u>Convolution theorem(1.2.2) : [16]</u>

The Laplace transform of the convolution

$$f(t)*g(t)=\int_0^t f(t-\tau)g(\tau)d\tau=\int_0^t f(\tau)g(t-\tau) \qquad (1-15)$$

of the two function $f(t)$ and $g(t)$, which are equal to zero for $t>0$, is equal to the product of the Laplace transform of those function :

$$L\{f(t)*g(t);s\}=F(s)G(s), \qquad (1-16)$$

Under the assumption that both $F(s)$ and $G(s)$ exist. We will use the property (1-16) for the evaluation of the Laplace transform of the Riemann-Liouville fractional integral.

Laplace transform of the fractional derivative(1.2.3) : [16]

Another useful property which we need is the formula for the Laplace transform of the derivative of an integer order n of the function $f(t)$;

$$L\{f^{n}(t);s\} = s^{n}F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = s^{n}F(s) - \sum_{k=0}^{n-1} s^{k} f^{(n-k-1)}(0), \qquad (1-17)$$

This can be obtained from the definition (1-14) by integrating by parts under the assumption that the corresponding integrals exist.

(1.3) Mittag-Leffler function: [16]

The exponential function, exp(z), plays a very important role in the theory of integer-order differential equations. Its one-parameter generalization, the function which is denoted by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k+1)} \quad , \qquad (1-18)$$

was introduced by G. M. Mittag-Leffler and studied also by A. Wiman.

Definition and relation to some other functions(1.3.1)

A two-parameter function of the mittag-Leffler type is defined by the series expansion

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad (\alpha>0,\beta>0) \qquad (1-19)$$

It follows from the definition (1-19) that

$$E_{1,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)}=\sum_{k=0}^{\infty}\frac{z^k}{k!}=\exp(z), \qquad (1-20)$$

$$E_{1,2}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+2)}=\sum_{k=0}^{\infty}\frac{z^k}{(k+1)!}=\frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{k+1}}{(k+1)!}=\frac{\exp(z)-1}{z}, \qquad (1-21)$$

$$E_{1,3}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+3)}=\sum_{k=0}^{\infty}\frac{z^k}{(k+2)!}=\frac{1}{z^2}\sum_{k=0}^{\infty}\frac{z^{k+2}}{(k+2)!}=\frac{\exp(z)-1-z}{z^2}, \qquad (1-22)$$

and in general

$$E_{1,m}(z)=\frac{1}{z^{m-1}}\{\exp(z)-\sum_{k=0}^{m-2}\frac{z^k}{k!}\}. \qquad (1-23)$$

(1.4) Gamma function: [16]

The gamma function $\Gamma(z)$ is defined by the integral

$$\Gamma(z)=\int_0^{\infty}\exp(-t)t^{z-1}\,dt, \qquad (1-24)$$

Which converges in the right half of the complex plane $\mathrm{Re}(z)>0$.

Indeed, we have

$$\Gamma(x+iy)=\int_0^\infty \exp(-t)\, t^{x-1+iy}\,dt=\int_0^\infty \exp(-t)\, t^{x-1}\exp(iy\log(t))\,dt$$

$$=\int_0^\infty \exp(-t)\, t^{x-1}[\cos(y\log(t))+i\sin(y\log(t))]\,dt. \qquad (1-25)$$

The expression in the square brackets in (1-25) is bounded for all t; Convergence at infinity is provided by $\exp(-t)$, and convergence at t=0 we must have $x=\mathrm{Re}(z)>1$.

(1.5) Definition of the fractional derivative: [12]

Let f be a function of class C and let $\mu>0$. Let m be the smallest integer that exceeds μ. Then the fractional derivative of f of order μ is defined as

$$D^{\mu} f(t)=D^{m}[D^{-v} f(t)], \qquad \mu>0, \quad t>0 \qquad (1-26)$$

(if it exists) where $v=m-\mu>0$.

(1.6) Riemann-Liouville fractional derivatives: [16]

The Reamann-Liouville fractional derivative is defined by the formula

$${}_aD_t^p f(t)=\left(\frac{d}{dt}\right)^{m+1}\int_a^t (t-\tau)^{m-p} f(\tau)\,d\tau, \quad (m\le p\le m+1). \qquad (1-27)$$

The expression (1-28) it is the most widely known definition of the fractional derivative; it is usually called the Riemann-Liouville definition.

(1.7)Fourier transform of the fractional derivative: [9]

The Fourier transform of a function $f:\Re \to C$, defined by

$$F[f](v) = f^{\wedge}(v) = \frac{1}{\sqrt{2\pi}} \int_R f(u)\exp(-i\,vu)du, \quad v \in \Re, \qquad (1-28)$$

is a powerful tool in the analysis of operators commuting with the translation operator. Its inverse is given by

$$f(x) = F^{-1}[f^{\wedge}(v)](x) = \frac{1}{\sqrt{2\pi}} \int_R f^{\wedge}(v)\exp(ixv)dv$$

For almost all $x \in \Re$ if f and $f^{\wedge}$ belong to $L^1(\Re)$ Two of the basic properties of the Fourier transform are

$$F[f^{\wedge}](v) = (iv)^n f^{\wedge}(v), \quad v \in \Re \qquad (1-29)$$

$$[F\,f]^n(v) = F[(-ix)^n f(x)](v), \qquad (1-30)$$

valid for sufficiently function f; (1-29) holds if, for example,

$$f \in L^1(\Re) \bigcap AC^{n-1}(\Re)$$

with $f^n \in L^1(\Re)$ while for (1-28) it is sufficient that f as well as $x^n f(x)$ belong to $L^1(\Re)$.

(1.8) Error functions: [7, 8]

The functions, denoted by erf , is define as

$$erf x = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt,$$

and

$$erfc\ x = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-t^2)\, dt, \qquad (1-31)$$

is known as the complementary error function.

Properties of the error function(1.8.1)

1. Relationships:

$$erf\ x + erf\,c\ x = 1, \quad erf(-x) = -erf\ x, \; erf\,c(-x) = 2 - erf\,c\ x.$$

2. Relationship with normal probability function:

$$\frac{1}{\sqrt{2\pi}} \int_0^x \exp(-\frac{1}{2}t^2)\ dt = \frac{1}{2}\ erf(\frac{x}{\sqrt{2}})\,.$$

Expansions(1.8.2)

1. Series expansions:

$$erf\ x = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)\, n!} = \frac{2}{\sqrt{\pi}}(x - \frac{x^3}{3} + \frac{1}{2!}\frac{x^5}{5} - \frac{1}{3!}\frac{x^7}{7} + \ldots\ldots)$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{3}{2}) \exp(-x^2)}{\Gamma(n+\frac{3}{2})}\, x^{2n+1} = \frac{2}{\sqrt{\pi}} \exp(-x^2)\,(x + \frac{2}{3}x^3 + \frac{4}{15}x^5 \ldots\ldots).$$

2. Asymptotic expansion: $For \quad z \to \infty, |\arg z| < \frac{3}{4}\pi,$

$$erf\,c \approx \frac{2}{\sqrt{\pi}} \frac{\exp(-z^2)}{2z} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{n!(2z)^{2n}}$$

$$\approx \frac{2}{\sqrt{\pi}} \frac{\exp(-z^2)}{2z} (1 - \frac{1}{2z^2} + \frac{3}{4z^4} - \frac{15}{8z^6} + \ldots\ldots).$$

The Repeated integrals of the error function have been investigated by Hartree (1936) who puts

$$i\,erfc\;x = 2\pi^{1/2}\;Erfc\;x \quad , \quad i^n erfc\;x = \int_x^{\infty} i^{n-1} erfc\;t\;dt\,.$$

(1.9) Magneto hydrodynamics [17]:

Magneto hydrodynamics (MHD) is the branch of continuum mechanics which deals with the motion of an electrically conducting fluid in the presence of a magnetic field. The subject is also some times called ‘hydrodynamics ‘or ‘magneto-fluid dynamics ‘. The motion of conducting material across the magnetic lines of force creates potential differences which, in general, cause electric currents to flow. The magnetic fields associated with these currents modify the magnetic field which creates them. In other words, the fluid flow alters the electromagnetic state of the system. On the other hand, the flow of electric current across a magnetic field is associated with a body force, the so-called Lorentz force, which influences the fluid flow. It is this intimate interdependence of hydrodynamics and electrodynamics which really defines and characterizes magnetohydrodynamics.

Chapter Two

Flow induced by constantly accelerating plate

Introduction

In this chapter, the flow induced by constantly accelerating plate is considered. It is found that the governing equations are controlled by many dimensionless numbers. The governing equation is solved by many Laplace and Fourier techniques. In the end of this chapter, the velocity field analyzed through plotting many graphing.

(2.1)Problem statement

Consideration is given to a conducting fluid permeated by an imposed magnetic field B_0 which acts in the positive y-direction. In the low-magnetic Reynolds number approximation, the magnetic body force is represented by $\sigma B_\circ^2 u$ .Consider an incompressible fractional Burgers' fluid lying over an infinitely extended plate which is situated in the (x,z) plane. Initially, the fluid is at rest and at time $t=0^+$, the infinite plate to slide in its own plane with a motion of the constant acceleration A. Owing to the shear, the fluid above the plate is gradually moved.

Under these considerations, the governing equation, in the absence of pressure gradient in the flow direction, is given by

$$(1+\lambda_1^\alpha D_t^\alpha+\lambda_2^\alpha D_t^{2\alpha})\frac{\partial u}{\partial t}=\upsilon(1+\lambda_3^\beta D_t^\beta)\frac{\partial^2 u}{\partial y^2}-M(1+\lambda_1^\alpha D_t^\alpha+\lambda_2^\alpha D_t^{2\alpha})u$$

Where $\upsilon=\frac{\mu}{\rho}$ is the kinematics' viscosity of the fluid and $M=\frac{\sigma B_\circ^2 u}{\rho}$.

The associated initial and boundary condition are follows:

Initial condition:

$$u(y,0)=\frac{\partial u(y,0)}{\partial t}=0\ ,\quad y>0$$

Boundary conditions:

$$u(0,t)=At\quad ,\quad t>0$$

Moreover, the natural conditions are

$$u(y,t)\ ,\frac{\partial u(y,t)}{\partial y}\to 0\ \ as\ \ y\to\infty\ \ and\ \ t>0$$

Have to be also satisfied. In order to solve this problem, we shall use the Fourier sine and Laplace transforms.

(2-2)Solution of the problem:

The constitutive equations for an incompressible fractional burgers' fluid are given by

$$T=-PI+S,(1+\lambda_1^\alpha\widetilde{D}_t^\alpha+\lambda_2^{2\alpha}\widetilde{D}_t^{2\alpha})S=\mu(1+\lambda_3^\beta\widetilde{D}_t^\beta)A \qquad (2-1)$$

Where T is the Cauchy stress tensor,-PI denotes the indeterminate spherical stress, S the extra stress tensor, $A=L+L^T$ the first Rivlin-Ericksen tensor, where L the velocity gradient, μ the dynamic viscosity of the of the fluid, λ_1 and $\lambda_3(<\lambda_1)$ the relaxation and retardation times, respectively, λ_2 is the new material parameter of the Burgers' fluid, α and β the fractional calculus parameters such that $0\le\alpha\le\beta\le 1$ and $\tilde{D}_t^p$ the upper connected fractional derivative defined by

$$\tilde{D}_t^p S = D_t^P S + v\cdot\nabla S - LS - SL^T ,$$
$$\tilde{D}_t^p A = \tilde{D}_t^p A + v\cdot\nabla A - LA - AL^T \qquad (2-2)$$

In which $D_t^p(=\partial_t^p)$ is the fractional differentiation operator of order p with respect to t and may be defined as [16]

$$D_t^p[f(t)] = \frac{1}{\Gamma(1-p)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(1-\tau)}d\tau , \qquad 0\le p\le 1 \qquad (2-3)$$

Here $\Gamma(.)$ denotes the Gamma function and

$$\tilde{D}_t^{2p} S = \tilde{D}_T^P(\tilde{D}_T^P S) , \qquad (2-4)$$

The equations of motion in absence of body force can be described as

$$\rho\frac{d\vec{v}}{dt} = \nabla\cdot\vec{T}, \qquad (2-5)$$

Where ρ is the density of the fluid and d/dt represents the material time derivative.

Since the fluid is incompressible, it can undergo only is isochoric motion and hence

$$\nabla\cdot\vec{v}=0, \qquad (2-6)$$

For the following problems of unidirectional flow, the intrinsic velocity field takes the form

$$\vec{v}=[u(y,t),0,0] \qquad (2-7)$$

Where $u(y,t)$ is the velocity in the x-coordinates direction. For this velocity field, the constraint of incompressibility (2-6) is automatically satisfied, we also assume that the extra stress S depends on y and t only. Substituting Eq. (2-7) into (2-1), (2-5) and taking account of the initial conditions $S(y,0)=\partial_t S(y,0)=0, \quad y>0$.

i.e. the fluid being at rest up to the time t = 0. For the components of the stress field S, we have $S_{yy}=S_{zz}=S_{xz}=S_{yz}=0$ and $S_{xy}=S_{yx}$, this yields

$$\rho\frac{d\vec{v}}{dt}=\nabla\cdot\vec{T}$$

$$\rho\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x}+\frac{\partial T_{xy}}{\partial x}+\frac{\partial T_{xy}}{\partial y},$$

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+\frac{\partial S_{xy}}{\partial y}, \qquad (2-8)$$

The equation of motion yields the following scalar equations:

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+\frac{\partial S_{xy}}{\partial y}-\sigma B_0^2 u \qquad (2-9)$$

Where ρ is the constant density of the fluid.

Now, Since

$$L=\begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} \quad , \quad L^T=\begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix} \quad , \quad S=\begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{bmatrix}$$

And, since

$$\vec{v}=[u(y,t),0,0],\ A=L+L^T,$$

$$S(y,0)=\partial_t S(y,0)=0, S_{yy}=S_{zz}=S_{xz}=S_{yz}=0, S_{xy}=S_{yx}.$$

Thus,

$$L=\begin{bmatrix} 0 & u_y & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, L^T=\begin{bmatrix} 0 & 0 & 0 \\ u_y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A=\begin{bmatrix} 0 & u_y & 0 \\ u_y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$LS=\begin{bmatrix} u_y S_{yx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, LS^T=\begin{bmatrix} u_y S_{xy} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, LA=\begin{bmatrix} (\frac{\partial u}{\partial y})^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$AL^T=\begin{bmatrix} (\frac{\partial u}{\partial y})^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, (\vec{v}\cdot\nabla)S=u\frac{\partial}{\partial x}\begin{bmatrix} S_{xx} & S_{xy} & 0 \\ S_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=0,$$

$$(\vec{v}\cdot\nabla)A = u\frac{\partial}{\partial x}\begin{bmatrix} o & u_y & 0 \\ u_y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0\,.$$

Now, Since

$$S + \lambda_1^{\alpha}\widetilde{D}_t^{\alpha}S + \lambda_2^{\alpha}\widetilde{D}_t^{2\alpha}S = \mu(A + \lambda_3^{\beta}\widetilde{D}_t^{\beta}A)$$

Thus,

$$S + \lambda_1^{\alpha}(D_t^{\alpha}S + (\vec{v}\cdot\nabla)S - LS - SL^T) + \lambda_2^{\alpha}D_t^{\alpha}(\vec{v}\cdot\nabla)S - LS - SL^T)$$
$$= \mu(A + \lambda_3^{\beta}(D_t^{\beta}A + (\vec{v}\cdot\nabla)A - LA - AL^T)) \qquad (2-10)$$

$$D_t^{\alpha}S + (\vec{v}\cdot\nabla)S - LS - SL^T$$
$$= D_t^{\alpha}\begin{bmatrix} S_{xx} & S_{xy} & 0 \\ S_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 0 - \begin{bmatrix} u_y S_{xy} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} u_y S_{xy} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} D_t^{\alpha}S_{xx} - 2u_y S_{xy} & D_t^{\alpha}S_{xy} & 0 \\ D_t^{\alpha}S_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (2-11)$$

$$D_t^{\alpha}(D_t^{2}S + (\vec{v}\cdot\nabla)S - LS - SL^T)$$
$$= D_t^{\alpha}(D_t^{2}\begin{bmatrix} S_{xx} & S_{xy} & 0 \\ S_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 0 + \begin{bmatrix} u_y S_{xy} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} u_y S_{xy} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= D_t^{\alpha}\begin{bmatrix} D_t^{2}S_{xx} - 2u_y S_{xy} & D_t^{2}S_{xy} & 0 \\ D_t^{2}S_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (2-12)$$

$$D_t^{\beta} A+(\vec{v}\cdot\nabla)A-LA-AL^T$$

$$=D_t^{\beta}\begin{bmatrix}0 & u_y & 0\\ u_y & 0 & 0\\ 0 & 0 & 0\end{bmatrix}+0-\begin{bmatrix}(\frac{\partial u}{\partial y})^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}-\begin{bmatrix}(\frac{\partial u}{\partial y})^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}-2(\frac{\partial u}{\partial y})^2 & D_t^{\beta} u_y & 0\\ D_t^{\beta} u_y & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \qquad (2-13)$$

$$\begin{bmatrix}S_{xx} & S_{xy} & 0\\ S_{xy} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}+\lambda_1^{\alpha}\begin{bmatrix}D_t^{\alpha} S_{xx}-2u_y S_{xy} & D_t^{\alpha} S_{xy} & 0\\ D_t^{\alpha} S_{xy} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}+\lambda_2^{\alpha} D_t^{\alpha}\begin{bmatrix}D_t^{2} S_{xx}-2u_y S_{xy} & D_t^{2} S_{xy} & 0\\ D_t^{2} S_{xy} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$=M\left(\begin{bmatrix}0 & u_y & 0\\ u_y & 0 & 0\\ 0 & 0 & 0\end{bmatrix}+\lambda_3^{\beta}\begin{bmatrix}-2(\frac{\partial u}{\partial y})^2 & D_t^{\beta} u_y & 0\\ D_t^{\beta} u_y & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right) \qquad (2-14)$$

Hence,

$$(1+\lambda_1^{\alpha} D_t^{\alpha}+\lambda_2^{\alpha} D_t^{2\alpha})S_{xx}=\mu(1+\lambda_3^{\beta} D_t^{\beta})\frac{\partial u}{\partial y} \qquad (2-15)$$

$$(1+\lambda_1^{\alpha} D_t^{\alpha}+\lambda_2^{\alpha} D_t^{2\varepsilon})S_{xx}-2S_{xy}[\lambda_1^{\alpha}+\lambda_2^{\alpha} D_t^{\alpha}]\frac{\partial u}{\partial y}-2\lambda_2^{\alpha}\frac{\partial u}{\partial y} D_t^{\alpha} S_{xy}=-2\mu\lambda_3^{\beta}\left(\frac{\partial u}{\partial y}\right)^2 \qquad (2-16)$$

Eliminating S_{xy} between Eqs. (2-9) and (2-15) ,we arrive at the following fractional differential equation

$$\rho(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\varepsilon})\frac{\partial u}{\partial t}=-(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\varepsilon})\frac{\partial p}{\partial x}+\mu(1+\lambda_3^{\beta}D_t^{\beta})\frac{\partial^2 u}{\partial y^2}$$
$$-(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})\sigma B_0^2 u \qquad (2-17)$$

The governing equation, in the absence of pressure gradient in the flow direction, is given by

$$(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})\frac{\partial u}{\partial t}=\upsilon(1+\lambda_3^{\beta}D_t^{\beta})\frac{\partial^2 u}{\partial y^2}-M(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})u \qquad (2-18)$$

Where $\upsilon=\frac{\mu}{\rho}$ is the kinematics' viscosity of the fluid and $M=\frac{\sigma B_{\circ}^2 u}{\rho}$.

The associated initial and boundary condition are follows:

Initial condition:

$$u(y,0)=\frac{\partial u(y,0)}{\partial t}=0\ ,\quad y>0 \qquad (2-19)$$

Boundary conditions:

$$u(0,t)=At\quad ,\quad t>0 \qquad (2-20)$$

Moreover, the natural conditions are

$$u(y,t)\,,\frac{\partial u(y,t)}{\partial y}\to 0\ \ as\ \ y\to\infty\ \ and\ \ t>0 \qquad (2-21)$$

Have to be also satisfied. In order to solve this problem, we shall use the Fourier sine and Laplace transforms.

Employing the non-dimensional quantities

$$U=\frac{u}{(\upsilon A)^{1/3}}\ ,\ \eta=y(\frac{A}{\upsilon^2})^{1/3}\ ,\ \tau=t(\frac{A^2}{\upsilon})^{1/3}\ ,\ \hat{\lambda}_1=\lambda_1(\frac{A^2}{\upsilon})^{1/3}\ ,$$

$$\hat{\lambda}_2=\lambda_2(\frac{A^4}{\upsilon^2})^{1/3}\quad and\ \hat{\lambda}_3=\lambda_3(\frac{A^2}{\upsilon})^{1/3}\qquad (2-22)$$

Eqs. (2-18) - (2-21) in dimensionless form are

$$(1+\lambda_1^\alpha D_t^\alpha+\lambda_2^\alpha D_t^{2\alpha})\frac{\partial U}{\partial\tau}=(1+\lambda_3^\beta D_t^\beta)\frac{\partial^2 U}{\partial\eta^2}-M(1+\lambda_1^\alpha D_t^\alpha+\lambda_2^\alpha D_t^{2\alpha})U\qquad (2-23)$$

$$U(\eta,0)=\frac{\partial U(\eta,0)}{\partial\tau}=\frac{\partial^2 U(\eta,0)}{\partial\tau^2}=0\ ,\ \eta>0\qquad (2-24)$$

$$U(0,\tau)=\tau\ \ ,\tau>0\qquad (2-25)$$

$$U(\eta,\tau)\ ,\ \frac{\partial U(\eta,\tau)}{\partial\eta}\to 0\ ,\ \ as\ \eta\to\infty\ and\ \tau\succ 0\qquad (2-26)$$

Where the dimensionless mark hat has been omitted for simplicity.

Now, applying Fourier sine transform [18] to Eqs. (2-23) and taking into account the boundary conditions (2-25) and (2-26), we find that

$$(1+\lambda_1^\alpha D_t^\alpha+\lambda_2^\alpha D_t^{2\alpha})\frac{\partial U_S(\xi,\tau)}{\partial\tau}=(1+\lambda_3^\beta D_t^\beta)\,(\sqrt{\frac{2}{\pi}}\,\zeta\tau-\xi^2 U_s(\xi,\tau))$$

$$-M(1+\lambda_1^\alpha D_t^\alpha+\lambda_2^\alpha D_t^{2\alpha})\,U_s(\xi,\tau)\qquad (2-27)$$

Where the Fourier sine transform $U_s(\xi,\tau) of U(\eta,t)$ has to satisfy the conditions

$$U_s(\xi,0)=\frac{\partial U_s(\xi,0)}{\partial\tau}=\frac{\partial^2 U_s(\xi,0)}{\partial\tau^2}=0\ ;\qquad \xi>0.\qquad (2-28)$$

Let $\overline{U}_s(\xi,s)$ be the Laplace transform of $U_s(\xi,\tau)$ defined by

$$\overline{U}_s(\xi,s)=\int_0^{\infty}U_s(\xi,\tau)\exp(-st)\,d\tau \quad , \; s>0. \qquad (2-29)$$

Taking the Laplace transform of Eq.(2-27), having in mind the initial conditions (2-28), we get

$$\overline{U}_s(\xi,s)=\sqrt{\frac{2}{\pi}}\frac{\xi(1+\lambda_3^{\beta}s^{\beta})}{s^2(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\xi^2+\xi^2\lambda_2^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{2\alpha}s^{2\alpha})} \qquad (2-30)$$

In order to obtain $U_s(\xi,\tau)=L^{-1}\{\overline{U}_s(\xi,s)\}$ with L^{-1} as the inverse Laplace transform operator and to avoid the lengthy procedure of residues and contour integral, we apply the discrete Laplace transform method. However, for a more suitable presentation of the final results, we rewrite Eq. (2-24) in the equivalent form

$$\overline{U}_s(\xi,s)=\sqrt{\frac{2}{\pi}}\frac{\xi(1+\lambda_3^{\beta}s^{\beta})(s+\xi^2)}{s^2(s+\xi^2)(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\xi^2+\xi^2\lambda_2^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{2\alpha}s^{2\alpha})}$$

$$=\sqrt{\frac{2}{\pi}}\frac{\xi(s+\xi^2)+\xi\lambda_3^{\beta}s^{\beta}(s+\xi^2)}{s^2(s+\xi^2)(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\xi^2+\xi^2\lambda_2^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{2\alpha}s^{2\alpha})}$$

$$=\sqrt{\frac{2}{\pi}}\frac{\xi s+\xi^3+\xi\lambda_3^{\beta}s^{\beta+1}+\xi^3\lambda_3^{\beta}s^{\beta}\pm\xi\lambda_1^{\alpha}s^{\alpha+1}\pm\xi\lambda_2^{\alpha}s^{2\alpha+1}\pm\xi M\pm\xi M\lambda_1^{\alpha}s^{\alpha}\pm\xi M\lambda_2^{\alpha}s^{2\alpha}}{s^2(s+\xi^2)(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\xi^2+\xi^2\lambda_2^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{2\alpha}s^{2\alpha})}$$

$$=\sqrt{\frac{2}{\pi}}\,(\xi(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{\alpha}s^{2\alpha})-s^2\xi(\lambda_1^{\alpha}s^{\alpha-1}+\lambda_2^{\alpha}s^{2\alpha-1}-\lambda_3^{\beta}s^{\beta-1}$$

$$+Ms^{-2}+M\lambda_1^{\alpha}s^{\alpha-2}+M\lambda_2^{\alpha}s^{2\alpha-2})\times\frac{1}{s^2(s+\xi^2)(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{\alpha}s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}\frac{\xi}{s^2(s+\xi^2)}-\frac{\xi(\lambda_1^{\alpha}s^{\alpha-1}+\lambda_2^{\alpha}s^{2\alpha-1}-\lambda_3^{\beta}s^{\beta-1}+Ms^{-2}+M\lambda_1^{\alpha}s^{\alpha-2}+M\lambda_2^{\alpha}s^{2\alpha-2})}{(s+\xi^2)(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{\alpha}s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}\frac{s+\xi^2-s}{\xi s^2(s+\xi^2)}-\frac{\xi(\lambda_1^{\alpha}s^{\alpha-1}+\lambda_2^{\alpha}s^{2\alpha-1}-\lambda_3^{\beta}s^{\beta-1}+Ms^{-2}+M\lambda_1^{\alpha}s^{\alpha-2}+M\lambda_2^{\alpha}s^{2\alpha-2})}{(s+\xi^2)(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{\alpha}s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}(\frac{1}{\xi s^2}-\frac{1}{\xi s(s+\xi^2)}-\frac{\xi(\lambda_1^{\alpha}s^{\alpha-1}+\lambda_2^{\alpha}s^{2\alpha-1}-\lambda_3^{\beta}s^{\beta-1}+Ms^{-2}+M\lambda_1^{\alpha}s^{\alpha-2}+M\lambda_2^{\alpha}s^{2\alpha-2})}{(s+\xi^2)(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{\alpha}s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}(\frac{1}{\xi s^2}-\frac{\xi^2}{\xi^3 s(s+\xi^2)}-\frac{\xi(\lambda_1^{\alpha}s^{\alpha-1}+\lambda_2^{\alpha}s^{2\alpha-1}-\lambda_3^{\beta}s^{\beta-1}+Ms^{-2}+M\lambda_1^{\alpha}s^{\alpha-2}+M\lambda_2^{\alpha}s^{2\alpha-2})}{(s+\xi^2)(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{\alpha}s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}(\frac{1}{\xi s^2}-(\frac{s+\xi^2-s}{s(s+\xi^2)})\frac{1}{\xi^3}-\frac{\xi(\lambda_1^{\alpha}s^{\alpha-1}+\lambda_2^{\alpha}s^{2\alpha-1}-\lambda_3^{\beta}s^{\beta-1}+Ms^{-2}+M\lambda_1^{\alpha}s^{\alpha-2}+M\lambda_2^{\alpha}s^{2\alpha-2})}{(s+\xi^2)(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{\alpha}s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}(\frac{1}{\xi s^2}-(\frac{1}{s}-\frac{1}{s+\xi^2})\frac{1}{\xi^3}$$

$$-\frac{\xi(\lambda_1^{\alpha}s^{\alpha-1}+\lambda_2^{\alpha}s^{2\alpha-1}-\lambda_3^{\beta}s^{\beta-1}+Ms^{-2}+M\lambda_1^{\alpha}s^{\alpha-2}+M\lambda_2^{\alpha}s^{2\alpha-2})}{(s+\xi^2)(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{\alpha}s^{2\alpha})}) \qquad (2-31)$$

Now, taking the part $\dfrac{1}{(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{\alpha}s^{2\alpha})}$

$$=\frac{1}{\lambda_1^{\alpha}(\frac{s}{\lambda_1^{\alpha}}+s^{\alpha+1}+\frac{\lambda_2^{\alpha}s^{2\alpha+1}}{\lambda_1^{\alpha}}+\frac{\xi^2}{\lambda_1^{\alpha}}+\frac{\xi^2\lambda_3^{\beta}s^{\beta}}{\lambda_1^{\alpha}}+\frac{M}{\lambda_1^{\alpha}}+Ms^{\alpha}+\frac{M\lambda_2^{\alpha}s^{2\alpha}}{\lambda_1^{\alpha}})}$$

And, by using $(\dfrac{1}{z+a}=\sum_{k=0}^{\infty}(-1)^k\dfrac{z^k}{a^{k+1}})$ we get

$$\frac{1}{\lambda_1^{\alpha}(\frac{s}{\lambda_1^{\alpha}}+s^{\alpha+1}+\frac{\lambda_2^{\alpha}s^{2\alpha+1}}{\lambda_1^{\alpha}}+\frac{\xi^2}{\lambda_1^{\alpha}}+\frac{\xi^2\lambda_3^{\beta}s^{\beta}}{\lambda_1^{\alpha}}+\frac{M}{\lambda_1^{\alpha}}+Ms^{\alpha}+\frac{M\lambda_2^{\alpha}s^{2\alpha}}{\lambda_1^{\alpha}})}$$

$$=\frac{1}{\lambda_1^{\alpha}}\sum_{m=0}^{\infty}(-1)^m\frac{(\frac{s}{\lambda_1^{\alpha}}+\frac{\lambda_2^{\alpha}s^{2\alpha+1}}{\lambda_1^{\alpha}}+\frac{\xi^2\lambda_3^{\beta}s^{\beta}}{\lambda_1^{\alpha}}+Ms^{\alpha}+\frac{M\lambda_2^{\alpha}s^{2\alpha}}{\lambda_1^{\alpha}})^m}{(s^{\alpha+1}+\frac{1}{\lambda_1^{\alpha}}(\xi^2+M))^{m+1}}$$

Now, taking the part $(\dfrac{s}{\lambda_1^{\alpha}}+\dfrac{\lambda_2^{\alpha}s^{2\alpha+1}}{\lambda_1^{\alpha}}+\dfrac{\xi^2\lambda_3^{\beta}s^{\beta}}{\lambda_1^{\alpha}}+Ms^{\alpha}+\dfrac{M\lambda_2^{\alpha}s^{2\alpha}}{\lambda_1^{\alpha}})^m$

$$=(\frac{s}{\lambda_1^{\alpha}})(1+\lambda_2^{\alpha}s^{2\alpha}+\xi^2\lambda_3^{\beta}s^{\beta-1}+M\lambda_1^{\alpha}s^{\alpha-1}+M\lambda_2^{\alpha}s^{2\alpha-1})^m$$

And, by using ($(1+b)^k = \sum_{m=0}^{k} \frac{k!b^m}{m!l!}$) we get

$$(\frac{s}{\lambda_1^\alpha})^m (1+\lambda_2^\alpha s^{2\alpha} + \xi^2 \lambda_3^\beta s^{\beta-1} + M\lambda_1^\alpha s^{\alpha-1} + M\lambda_2^\alpha s^{2\alpha-1})^m$$

$$= \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} (\lambda_2^\alpha s^{2\alpha} + \xi^2 \lambda_3^\beta s^{\beta-1} + M\lambda_1^\alpha s^{\alpha-1} + M\lambda_2^\alpha s^{2\alpha-1})^l$$

$$= (\frac{s}{\lambda_1^\alpha})^m \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} (\lambda_2^\alpha s^{2\alpha})^l (1+\frac{\xi^2 \lambda_3^\beta s^{\beta-1-2\alpha}}{\lambda_2^\alpha} + \frac{M\lambda_1^\alpha s^{-\alpha-1}}{\lambda_2^\alpha} + Ms^{-1})^l$$

$$= (\frac{s}{\lambda_1^\alpha})^m \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} (\lambda_2^\alpha s^{2\alpha})^l \sum_{j=0}^{l} \frac{l!}{j!(l-j)!} (\frac{\xi^2 \lambda_3^\beta s^{\beta-1-2\alpha}}{\lambda_2^\alpha} + \frac{M\lambda_1^\alpha s^{-\alpha-1}}{\lambda_2^\alpha} + Ms^{-1})^j$$

$$= (\frac{s}{\lambda_1^\alpha})^m \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} (\lambda_2^\alpha s^{2\alpha})^l \sum_{j=0}^{l} \frac{l!}{j!(l-j)!} (\frac{M}{s})^l (1+\frac{\xi^2 \lambda_3^\beta s^{\beta-2\alpha}}{M\lambda_2^\alpha} + \frac{\lambda_1^\alpha s^{-\alpha}}{\lambda_2^\alpha})^j$$

$$= (\frac{s}{\lambda_1^\alpha})^m \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} (\lambda_2^\alpha s^{2\alpha})^l \sum_{j=0}^{l} \frac{l!}{j!(l-j)!} (\frac{M}{s})^l \sum_{i=0}^{j} \frac{j!}{i!(j-i)!} (\frac{\xi^2 \lambda_3^\beta s^{\beta-2\alpha}}{M\lambda_2^\alpha} + \frac{\lambda_1^\alpha s^{-\alpha-1}}{\lambda_2^\alpha})^i$$

$$= (\frac{s}{\lambda_1^\alpha})^m \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} (\lambda_2^\alpha s^{2\alpha})^l \sum_{j=0}^{l} \frac{l!}{j!(l-j)!} (\frac{M}{s})^l \sum_{i=0}^{j} \frac{j!}{i!(j-i)!} (\frac{\lambda_1^\alpha s^{-\alpha}}{\lambda_2^\alpha})^i (1+\frac{\xi^2 \lambda_3^\beta s^{\beta-2\alpha}}{M\lambda_1^\alpha})^i$$

$$= (\frac{s}{\lambda_1^\alpha})^m \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} (\lambda_2^\alpha s^{2\alpha})^l \sum_{j=0}^{l} \frac{l!}{j!(l-j)!} (\frac{M}{s})^l \sum_{i=0}^{j} \frac{j!}{i!(j-i)!} (\frac{\lambda_1^\alpha s^{-\alpha}}{\lambda_2^\alpha})^i \sum_{d=0}^{i} \frac{i!}{d!(i-d)!} (\frac{\xi^2 \lambda_3^\beta s^{\beta-2\alpha}}{M\lambda_1^\alpha})^d$$

Hence, the Eq. (2-25) can be written under the form of a series as

$$\overline{U}_s(\xi,s)=\sqrt{\frac{2}{\pi}}\,\{[\frac{1}{\xi s^2}-(\frac{1}{s}-\frac{1}{s+\xi^2})\frac{1}{\xi^3}]-\xi(\lambda_1^{\alpha}s^{\alpha-1}+\lambda_2^{\alpha}s^{2\alpha-1}-\lambda_3^{\beta}s^{\beta-1}+Ms^{-2}$$
$$+M\lambda_1^{\alpha}s^{\alpha-2}+M\lambda_2^{\alpha}s^{2\alpha-2})\sum_{m=0}^{\infty}(-1)^m\,\frac{\sum_{l=0}^{m}\frac{1}{l!(m-l)!}\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}}{(s+\xi^2)(s^{\alpha+1}+\frac{1}{\lambda_1^{\alpha}}(\xi^2+M))^{m+1}}\}$$
$$\lambda_1^{\alpha(-m+i-d-1)}\;\lambda_2^{\alpha(l-i)}\;\lambda_3^{\beta d}\;M^{j-d}\;\xi^{2d}\;m!\,s^{\delta} \qquad (2-32)$$

In which $\delta=m+2\alpha l-j-\alpha i+\beta d-\alpha d.$

Now, applying the inversion formula term by term for the Laplace transform, Eq.(2-32) yields

$$U_s(\xi,\tau)=\sqrt{\frac{2}{\pi}}\,[\frac{\tau}{\xi}-\frac{1}{\xi^3}(1-\exp(-\xi^2\tau)]$$

$$-\int_0^{\tau}\xi\sqrt{\frac{2}{\pi}}\sum_{m=0}^{\infty}(-1)^m\sum_{l=0}^{m}\frac{1}{l!(m-l)!}\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}$$

$$\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}\,\lambda_1^{\alpha(-m+i-d-1)}\;\lambda_2^{\alpha(l-i)}\;\lambda_3^{\beta d}$$

$$M^{\,j-d}\;\xi^{2d}\times[\lambda_1^{\alpha}\sigma^{(\alpha+1)m+(2-\delta)-1}\;E^{(m)}_{(\alpha+1),(2-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})$$

$$+\lambda_2^{\alpha}\;\sigma^{(\alpha+1)m+(2-\alpha-\delta)-1}\;E^{(m)}_{(\alpha+1),(2-\alpha-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\;\sigma^{\alpha+1})$$

$$-\lambda_3^{\beta}\;\sigma^{(\alpha+1)m+(2+\alpha-\beta-\delta)-1}\;E^{(m)}_{(\alpha+1),(2+\alpha-\beta-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})$$

$$+M\;\sigma^{(\alpha+1)m+(3-\delta+\alpha)-1}\;E^{(m)}_{(\alpha+1),(3-\delta+\alpha)}(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})$$

$$+M\lambda_1^{\alpha}\sigma^{(\alpha+1)m+(3-\delta)-1}\;E^{(m)}_{(\alpha+1),(3-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})$$

$$+M\lambda_2^{\alpha}\sigma^{(\alpha+1)m+(3-\alpha-\delta)-1}\;E^{(m)}_{(\alpha+1),(3-\alpha-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})]$$

$$*\exp(-\xi^2(\tau-\sigma)]\,d\sigma \qquad (2-33)$$

Where "$*$" represents the convolution of two functions and

$$E_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\lambda n+\mu)},\quad \lambda,\mu>0, \qquad (2-34)$$

Denotes the generalized Mittag-Leffler function with

$$E^{(k)}_{\lambda,\mu}(z)=\frac{d^k}{dz^k}\,E_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{(n+k)!\,z^n}{n!\,\Gamma(\lambda n+\lambda k+\mu)}. \qquad (2-35)$$

Here, we used the following property of the generalized Mittad-Leffler function[16]

$$L^{-1}\{\frac{k!s^{\lambda-\mu}}{(s^{\lambda}\mp c)^{k+1}}\}=t^{\lambda k+\mu-1}E_{\lambda,\mu}^{(k)}(\mp ct^{\lambda}),\quad (\mathrm{Re}(s)\succ|c|^{1/\lambda}). \qquad (2-36)$$

Finally, inverting (2-33) by the Fourier transform [10]we find for the velocity $U(\xi,\tau)$ the expression

$$U(\eta,\tau)=U_N(\eta,\tau)-\frac{2}{\pi}\int_0^{\infty}\int_0^{\tau}\frac{\sin(\xi\eta)}{\xi}\sum_{m=0}^{\infty}(-1)^m\sum_{l=0}^{m}\frac{1}{l!(m-l)!}$$

$$\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}\lambda^{\alpha(-m+i-d-1)}\ \lambda_2^{\alpha(l-i)}\ \lambda_3^{\beta d}\ M^{j-d}$$

$$\xi^{2d}\times[\lambda_1^{\alpha}\sigma^{(\alpha+1)m+(2-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(2-\delta))}$$

$$+\lambda_2^{\alpha}\sigma^{(\alpha+1)m+(2-\alpha-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(2-\alpha-\delta)}$$

$$+\lambda_3^{\beta}\sigma^{(\alpha+1)m+(2+\alpha-\beta-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(2+\alpha-\beta-\delta))}$$

$$+M\sigma^{(\alpha+1)m+(3-\delta+\alpha)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(3-\delta+\alpha))}$$

$$+M\lambda_1^{\alpha}\sigma^{(\alpha+1)m+(3-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(3-\delta))}$$

$$+M\lambda_2^{\alpha}\sigma^{(\alpha+1)m+(3-\alpha-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(3-\alpha-\delta))}]$$

$$*\exp(-\xi^2(\tau-\sigma)d\sigma d\xi \qquad (2-37)$$

Whence,

$$U_N(\eta,\tau)=\tau-\frac{2}{\pi}\int_0^\infty(1-\exp(-\xi^2\tau)\frac{\sin(\xi\eta)}{\xi^3}d\xi=4\tau i^2\,Erfc(\frac{\eta}{2\sqrt{\tau}}), \qquad (2-38)$$

Represents the velocity field corresponding to a Newtonian fluid performing the same motion.

In the above relation $i^n Erfc(.)$ are the integrals of the complementary error function of Gauss.

(2-3) Results and discussion:

This section displays the graphical illustration velocity field for the flows analyzed in this investigation. We interpret these results with respect to the variation of emerging parameters of interest. The exact analytical solutions for accelerated flows have been obtained for a Burgers' fluid and a comparison is mad with the results for those of the fractional Oldroyd-B fluid.

Fig. (2-1) is prepared to show the effects of non-integer fractional parameters α on the velocity field, as well as a comparison between the fractional Oldroyd-B fluid and fractional Burgers' fluid for fixed values of other parameters. As seen from these figures that for time $\tau = 0.5$ the smaller the α, the more speedily the velocity decays for both the fluids. Moreover, for time $\tau = 0.5$ the velocity profiles for an Oldroyd-B fluid are greater than those for a Burgers' fluid. Its also observed that for time $\tau = 0.5$ the velocity profiles for Burgers' fluids approach the velocity profile of the fractional Oldroyd-B fluid and after some time it will become the same. Thus, it's obvious that the relaxation and retardation times and the orders of the fractional parameters have strong effects on the velocity field.

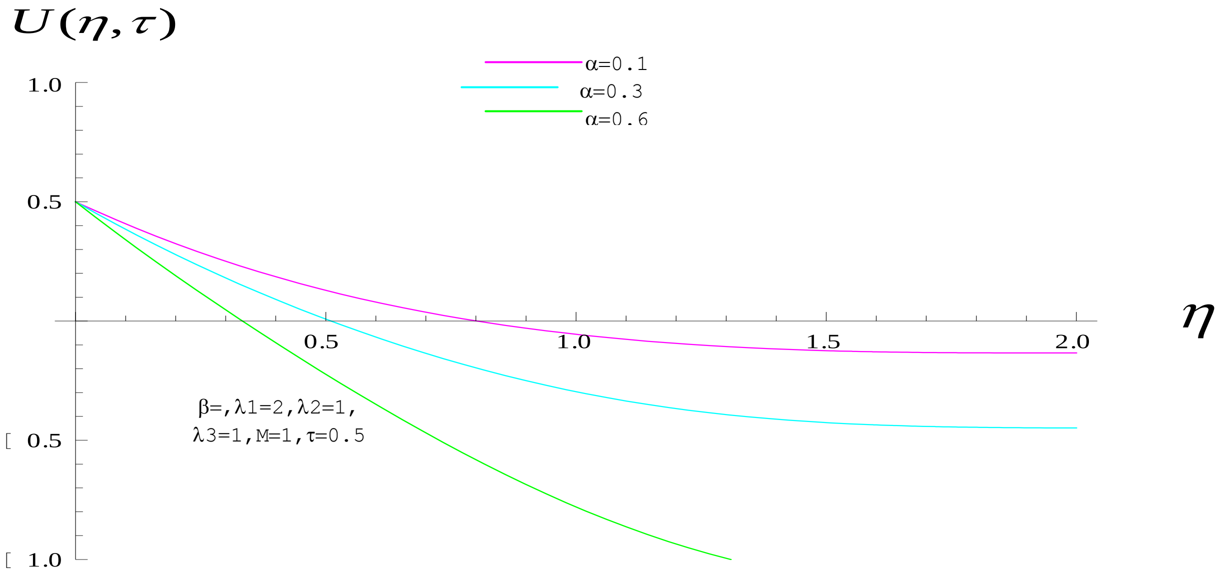
Fig. (2-2) is prepared to show the effects of non-integer fractional parameters β on the velocity field, as well as a comparison between the fractional Oldroyd-B fluid and fractional Burgers' fluid for fixed values of other parameters. It is observed that for time $\tau = 0.5$ the velocity will increase by the decreases in the parameter β.

It's also observed that for time $\tau = 0.5$ the velocity profiles for Burgers' fluids approach the velocity profile of the fractional Oldroyd-B fluid and after some time it will become the same.

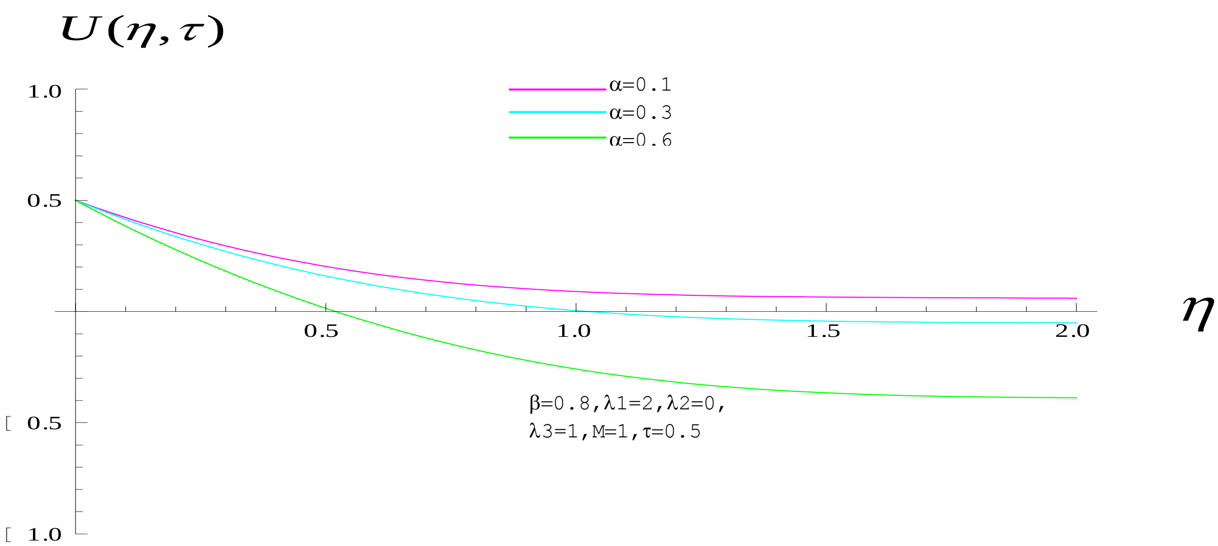
Fig. (2-3) shows the effects of new material parameter on the velocity field for fixed values of other parameters. It is observed that for time $\tau = 1$ the velocity will decrease by the increase in new material parameter λ_2.

Fig. (2-4) shows the variation of time τ on the velocity field for fixed values of other parameters. It's observed that the velocity will increase by the increase in time and after some time it will become the same.

Fig. (2-5) shows the velocity changes with the fractional parameters and the magnetic field parameter. It is observed that for $\alpha \le 0.2$ the velocity will decrease by the increase in the magnetic field M. However, one can see that an increase in the magnetic field M for $\alpha \le 0.6$ has quite the opposite effect to that of $\alpha \le 0.2$.

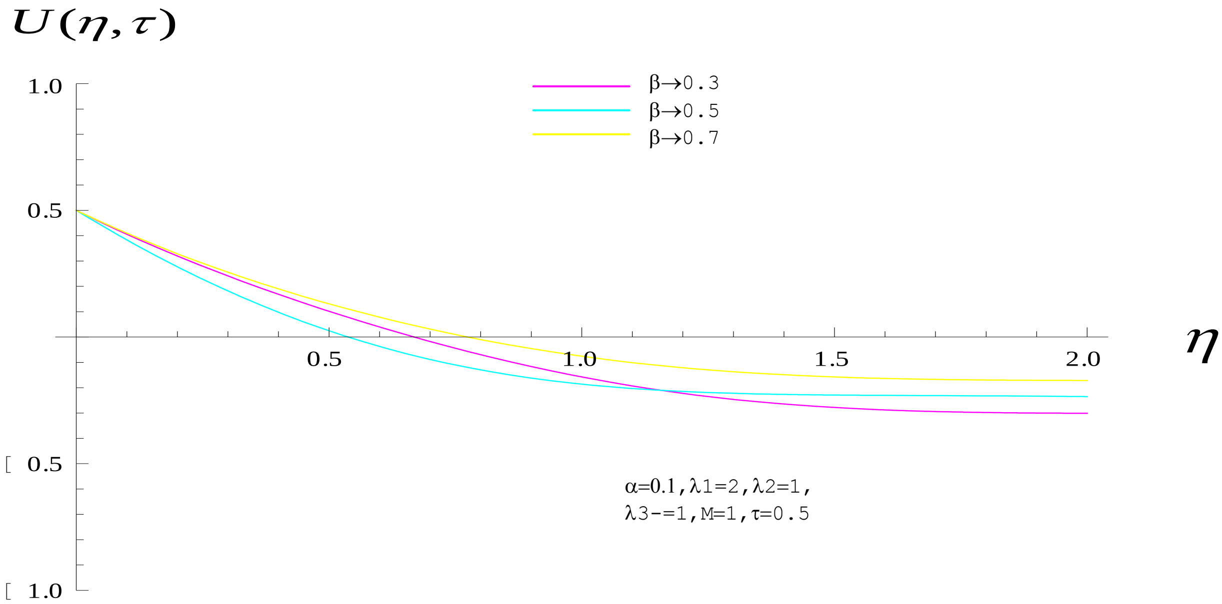


a) Burgers' model

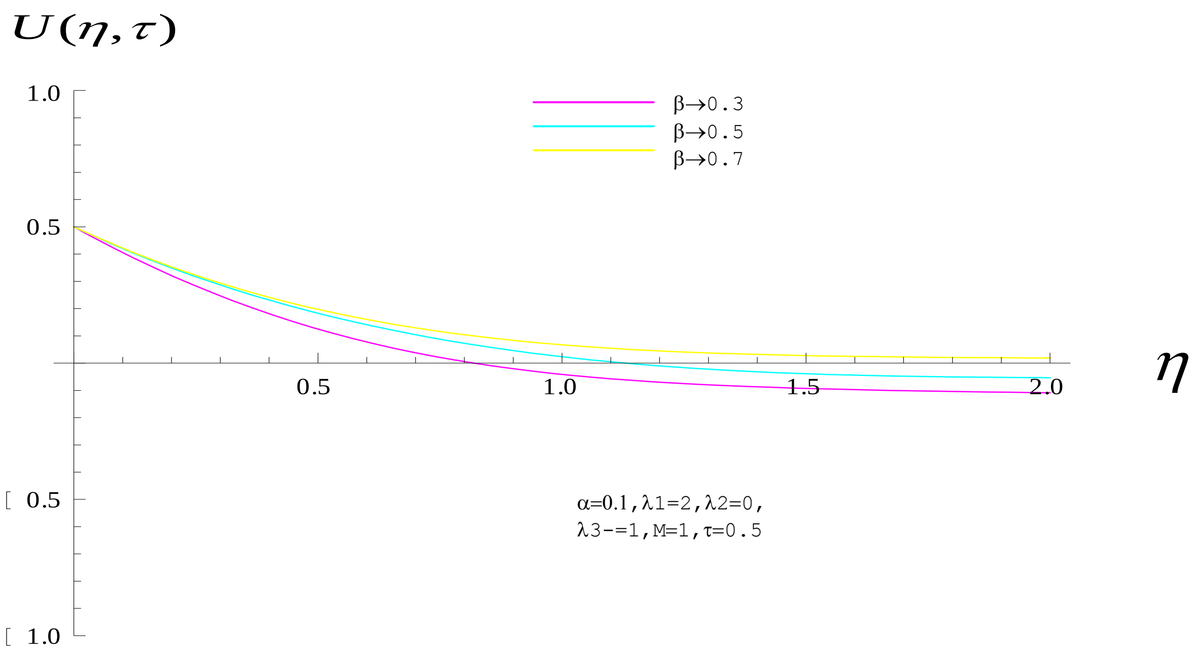


b) Oldroyd-B fluid

Fig.(2-1): Velocity $U(\eta,\tau)$ versus η for different values of α when other parameters are fixed.



a)Burgers' model



b)Oldroyd-B fluid

Fig.(2-2): Velocity $U(\eta,\tau)$ versus η for different values of β when other parameters are fixed.

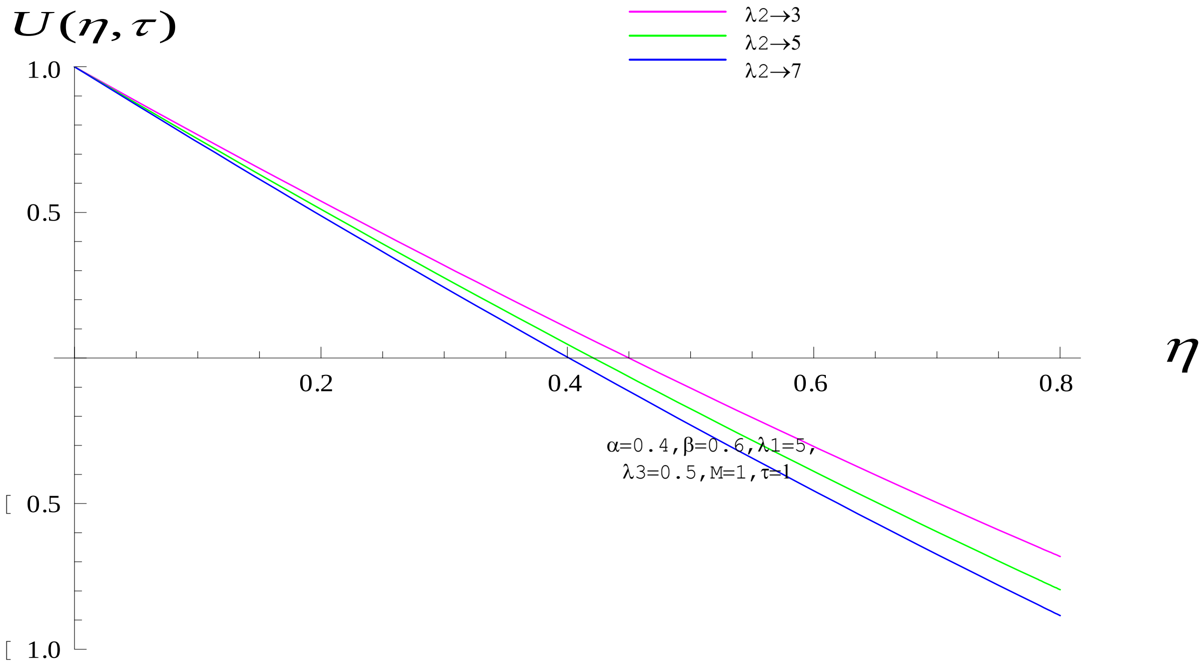


Fig.(2-3): Velocity $U(\eta,\tau)$ versus η for different values of λ_2 when other parameters are fixed.

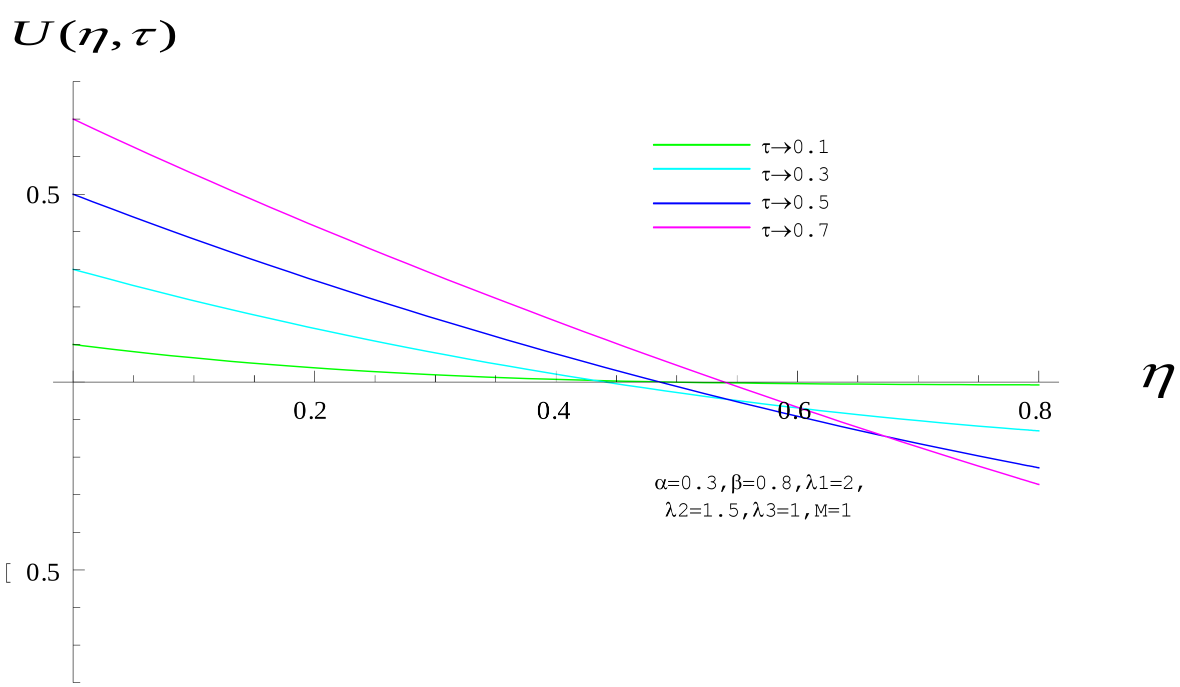


Fig.(2-4): Velocity $U(\eta,\tau)$ versus η for different values of τ when other parameters are fixed.

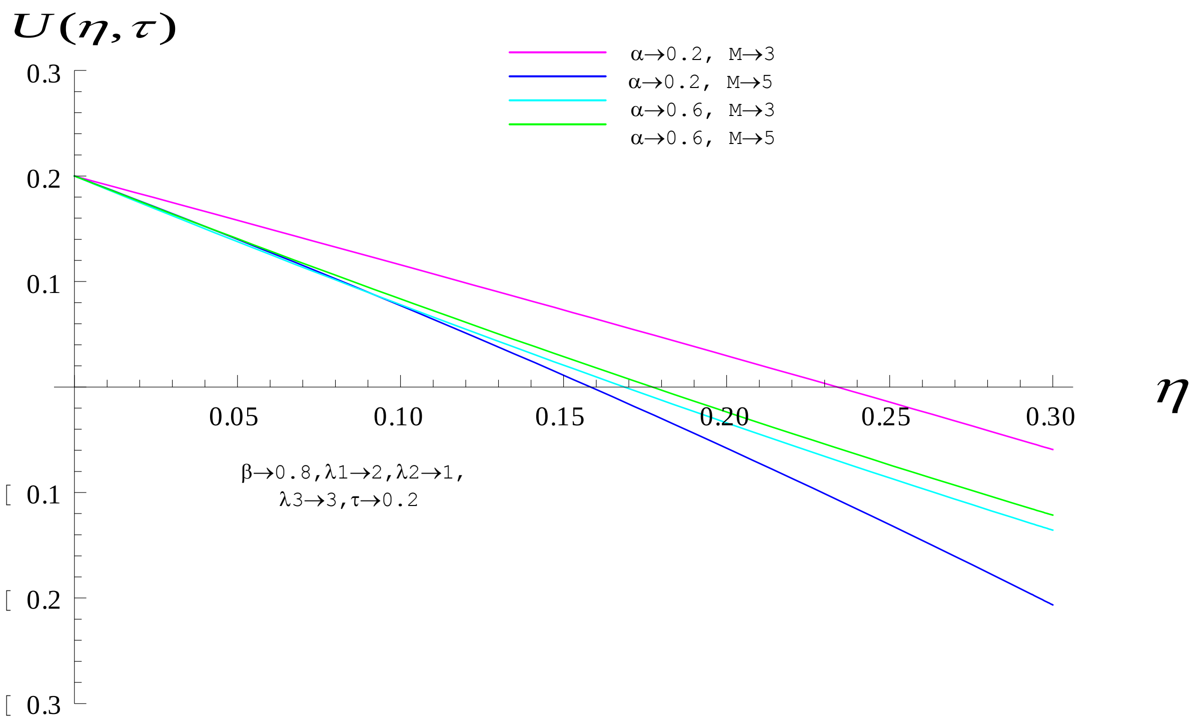


Fig.(2-5): Velocity $U(\eta,\tau)$ versus η for different values of α, M when other parameters are fixed.

Chapter Three

Flow induced by variable accelerating plate

Introduction

In this chapter, the flow induced by variable accelerating plate is considered. It is found that the governing equations are controlled by many dimensionless numbers. The governing equation is solved by many Laplace and Fourier techniques. In the end of this chapter, the velocity field analyzed through plotting many graphing.

(2.1)Problem statement:

Consideration is given to a conducting fluid permeated by an imposed magnetic field B_0 which acts in the positive y- direction. In the low-magnetic Reynolds number approximation, the magnetic body force is represented by $\sigma B_\circ^2 u$ .Consider an incompressible fractional Burgers' fluid lying over an infinitely extended plate which is situated in the (x,z) plane. Initially, the fluid is at rest and at time $t = 0^+$, the infinite plate to slide in its own plane with a motion of the variable acceleration A. Owing to the shear, the fluid above the plate is gradually moved. Under these considerations, the governing equation, in the absence of pressure gradient in the flow direction, is given by

$$(1+\lambda_1^\alpha D_t^\alpha+\lambda_2^\alpha D_t^{2\alpha})\frac{\partial u}{\partial t}=\upsilon(1+\lambda_3^\beta D_t^\beta)\frac{\partial^2 u}{\partial y^2}-M(1+\lambda_1^\alpha D_t^\alpha+\lambda_2^\alpha D_t^{2\alpha})u$$

Where $\upsilon=\frac{\mu}{\rho}$ is the kinematics' viscosity of the fluid and $M=\frac{\sigma B_\circ^2 u}{\rho}$.

The associated initial and boundary condition are follows:

Initial condition:

$$u(y,0)=\frac{\partial u(y,0)}{\partial t}=0\ ,\quad y>0$$

Boundary conditions:

$$u(0,t)=Bt^2\quad ,\quad t>0$$

Moreover, the natural conditions are

$$u(y,t)\ ,\frac{\partial u(y,t)}{\partial y}\to 0\ \ as\ \ y\to\infty\ \ and\ \ t>0$$

Have to be also satisfied. In order to solve this problem, we shall use the Fourier sine and Laplace transforms.

<u>(3.1)Solution of problem:</u>

By using the same procedure as in chapter two, the motion equation can be written as:

$$(1+\lambda_1^\alpha D_t^\alpha+\lambda_2^\alpha D_t^{2\alpha})S_{xx}=\mu(1+\lambda_3^\beta D_t^\beta)\frac{\partial u}{\partial y}\qquad(3-1)$$

$$(1+\lambda_1^\alpha D_t^\alpha+\lambda_2^\alpha D_t^{2\varepsilon})S_{xx}-2S_{xy}[\lambda_1^\alpha+\lambda_2^\alpha D_t^\alpha]\frac{\partial u}{\partial y}-2\lambda_2^\alpha\frac{\partial u}{\partial y}D_t^\alpha S_{xy}=-2\mu\lambda_3^\beta\left(\frac{\partial u}{\partial y}\right)^2\qquad(3-2)$$

And the governing equation, in the absence of pressure gradient in the flow direction, is given by

$$(1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha})\frac{\partial u}{\partial t} = \upsilon(1+\lambda_3^{\beta} D_t^{\beta})\frac{\partial^2 u}{\partial y^2} - M(1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha})u \qquad (3-3.18)$$

Where $\upsilon = \frac{\mu}{\rho}$ is the kinematics' viscosity of the fluid and $M = \frac{\sigma B_\circ^2 u}{\rho}$.

The associated initial and boundary condition are follows:

Initial condition:

$$u(y,0) = \frac{\partial u(y,0)}{\partial t} = 0\,, \quad y > 0 \qquad (3-4)$$

Boundary conditions:

$$u(0,t) = Bt^2, \quad t > 0 \qquad (3-5)$$

Moreover, the natural conditions are

$$u(y,t)\,, \frac{\partial u(y,t)}{\partial y} \to 0 \;\; as \;\; y \to \infty \;\; and \;\; t > 0 \qquad (3-6)$$

Here the governing problem can be normalized using the following dimensionless

$$U = \frac{u}{(\upsilon^2 B)^{1/5}}\,,\ \eta = y(\frac{B}{\upsilon^3})^{1/5}\,,\ \tau = t(\frac{B^2}{\upsilon})^{1/5}\,,\ \hat{\lambda}_1 = \lambda_1(\frac{B^2}{\upsilon})^{1/5}\,,$$

$$\hat{\lambda}_2 = \lambda_2(\frac{B^4}{\upsilon^2})^{1/5} \quad and \;\; \hat{\lambda}_3 = \lambda_3(\frac{B^2}{\upsilon})^{1/5} \qquad (3-7)$$

Eqs. (3-3) - (3-6) in dimensionless form are

$$(1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha})\frac{\partial U}{\partial \tau} = (1+\lambda_3^{\beta} D_t^{\beta})\frac{\partial^2 U}{\partial \eta^2} - M(1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha})U \qquad (3-8)$$

$$U(\eta,0)=\frac{\partial U(\eta,0)}{\partial \tau}=\frac{\partial^2 U(\eta,0)}{\partial \tau^2}=0\,,\,\eta>0 \qquad (3-9)$$

$$U(0,\tau)=\tau^2 \quad ,\tau>0 \qquad (3-10)$$

$$U(\eta,\tau)\,,\,\frac{\partial U(\eta,\tau)}{\partial \eta}\to 0\,, \quad as\ \eta\to\infty\ and \quad \tau>0 \qquad (3-11)$$

Where the dimensionless mark hat has been omitted for simplicity.

Now, applying Fourier sine transform [18] to Eqs. (3-8) and taking into account the boundary conditions (3-10) and (3-11), we find that

$$(1+\lambda_1^\alpha D_t^\alpha+\lambda_2^\alpha D_t^{2\alpha})\frac{\partial U_s(\xi,\tau)}{\partial \tau}=(1+\lambda_3^\beta D_t^\beta)\,(\sqrt{\frac{2}{\pi}}\,\zeta\tau^2-\xi^2 U_s(\xi,\tau))$$
$$-M(1+\lambda_1^\alpha D_t^\alpha+\lambda_2^\alpha D_t^{2\alpha})\,U_s(\xi,\tau) \qquad (3-12.27)$$

Where the Fourier sine transform $U_s(\xi,\tau) of U(\eta,t)$ has to satisfy the conditions

$$U_s(\xi,0)=\frac{\partial U_s(\xi,0)}{\partial \tau}=\frac{\partial^2 U_s(\xi,0)}{\partial \tau^2}=0\,; \quad \xi>0. \qquad (3-13.28)$$

Let $\overline{U}_s(\xi,s)$ be the Laplace transform of $U_s(\xi,\tau)$ defined by

$$\overline{U}_s(\xi,s)=\int_0^\infty U_s(\xi,\tau)\exp(-st)\,d\tau \quad ,\ s>0. \qquad (3-14.29)$$

Taking the Laplace transform of Eq.(3-12), having in mind the initial conditions (3-13), we get

$$\overline{U}_s(\xi,s)=\sqrt{\frac{2}{\pi}}\frac{\xi(1+\lambda_3^\beta s^\beta)}{s^3(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_2^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^{2\alpha}s^{2\alpha})} \qquad (3-15)$$

In order to obtain $U_s(\xi,\tau) = L^{-1}\{\overline{U}_s(\xi,s)\}$ with L^{-1} as the inverse Laplace transform operator and to avoid the lengthy procedure of residues and contour integral, we apply the discrete Laplace transform method. However, for a more suitable presentation of the final results, we rewrite Eq. (3-15) in the equivalent form

$$\overline{U}_s(\xi,s) = \sqrt{\frac{2}{\pi}}\,\frac{\xi(1+\lambda_3^\beta s^\beta)(s+\xi^2)}{s^3(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_2^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^{2\alpha}s^{2\alpha})}$$

$$=\sqrt{\frac{2}{\pi}}\,\frac{\xi(s+\xi^2)+\xi\lambda_3^\beta s^\beta(s+\xi^2)}{s^3(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_2^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^{2\alpha}s^{2\alpha})}$$

$$=\sqrt{\frac{2}{\pi}}\,\frac{\xi s+\xi^3+\xi\lambda_3^\beta s^{\beta+1}+\xi^3\lambda_3^\beta s^\beta \pm \xi\lambda_1^\alpha s^{\alpha+1} \pm \xi\lambda_2^\alpha s^{2\alpha+1} \pm \xi M \pm \xi M\lambda_1^\alpha s^\alpha \pm \xi M\lambda_2^\alpha s^{2\alpha}}{s^3(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_2^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^{2\alpha}s^{2\alpha})}$$

$$=\sqrt{\frac{2}{\pi}}\,(\xi(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^\alpha s^{2\alpha})-s^3\xi(\lambda_1^\alpha s^{\alpha-2}+\lambda_2^\alpha s^{2\alpha-2}-\lambda_3^\beta s^{\beta-2}$$

$$+Ms^{-3}+M\lambda_1^\alpha s^{\alpha-3}+M\lambda_2^\alpha s^{2\alpha-3})\times\frac{1}{s^3(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^\alpha s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}\,\frac{\xi}{s^3(s+\xi^2)}-\frac{\xi(\lambda_1^\alpha s^{\alpha-2}+\lambda_2^\alpha s^{2\alpha-2}-\lambda_3^\beta s^{\beta-2}+Ms^{-3}+M\lambda_1^\alpha s^{\alpha-3}+M\lambda_2^\alpha s^{2\alpha-3})}{(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^\alpha s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}\frac{s+\xi^2-s}{\xi s^3(s+\xi^2)}-\frac{\xi(\lambda_1^\alpha s^{\alpha-2}+\lambda_2^\alpha s^{2\alpha-2}-\lambda_3^\beta s^{\beta-2}+Ms^{-3}+M\lambda_1^\alpha s^{\alpha-3}+M\lambda_2^\alpha s^{2\alpha-3})}{(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^\alpha s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}(\frac{1}{\xi s^3}-\frac{1}{\xi s^2(s+\xi^2)}-\frac{\xi(\lambda_1^\alpha s^{\alpha-2}+\lambda_2^\alpha s^{2\alpha-2}-\lambda_3^\beta s^{\beta-2}+Ms^{-3}+M\lambda_1^\alpha s^{\alpha-3}+M\lambda_2^\alpha s^{2\alpha-3})}{(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^\alpha s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}(\frac{1}{\xi s^3}-\frac{\xi^2}{\xi^3 s^2(s+\xi^2)}-\frac{\xi(\lambda_1^\alpha s^{\alpha-2}+\lambda_2^\alpha s^{2\alpha-2}-\lambda_3^\beta s^{\beta-2}+Ms^{-3}+M\lambda_1^\alpha s^{\alpha-3}+M\lambda_2^\alpha s^{2\alpha-3})}{(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^\alpha s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}(\frac{1}{\xi s^3}-(\frac{s+\xi^2-s}{\xi^3 s^2(s+\xi^2)})-\frac{\xi(\lambda_1^\alpha s^{\alpha-2}+\lambda_2^\alpha s^{2\alpha-2}-\lambda_3^\beta s^{\beta-2}+Ms^{-3}+M\lambda_1^\alpha s^{\alpha-3}+M\lambda_2^\alpha s^{2\alpha-3})}{(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^\alpha s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}(\frac{1}{\xi s^3}-\frac{1}{\xi^3 s^2}-\frac{1}{\xi^3 s(s+\xi^2)}$$
$$-\frac{\xi(\lambda_1^\alpha s^{\alpha-2}+\lambda_2^\alpha s^{2\alpha-2}-\lambda_3^\beta s^{\beta-2}+Ms^{-3}+M\lambda_1^\alpha s^{\alpha-3}+M\lambda_2^\alpha s^{2\alpha-3})}{(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^\alpha s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}(\frac{1}{\xi s^3}-\frac{1}{\xi^3 s^2}-\frac{\xi^2}{\xi^5 s(s+\xi^2)}$$
$$-\frac{\xi(\lambda_1^\alpha s^{\alpha-2}+\lambda_2^\alpha s^{2\alpha-2}-\lambda_3^\beta s^{\beta-2}+Ms^{-3}+M\lambda_1^\alpha s^{\alpha-3}+M\lambda_2^\alpha s^{2\alpha-3})}{(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^\alpha s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{1}{\xi s^{3}}-\frac{1}{\xi^{3}s^{2}}+\left(\frac{s+\xi^{2}-s}{s(s+\xi^{2})}\right)\frac{1}{\xi^{5}}\right.$$
$$\left.-\frac{\xi(\lambda_1^{\alpha}s^{\alpha-2}+\lambda_2^{\alpha}s^{2\alpha-2}-\lambda_3^{\beta}s^{\beta-2}+Ms^{-3}+M\lambda_1^{\alpha}s^{\alpha-3}+M\lambda_2^{\alpha}s^{2\alpha-3})}{(s+\xi^{2})(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\xi^{2}+\xi^{2}\lambda_3^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{\alpha}s^{2\alpha})}\right)$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{1}{\xi s^{3}}-\frac{1}{\xi^{3}s^{2}}+\left(\frac{s+\xi^{2}}{s(s+\xi^{2})}-\frac{s}{s(s+\xi^{2})}\right)\frac{1}{\xi^{5}}\right.$$
$$\left.-\frac{\xi(\lambda_1^{\alpha}s^{\alpha-2}+\lambda_2^{\alpha}s^{2\alpha-2}-\lambda_3^{\beta}s^{\beta-2}+Ms^{-3}+M\lambda_1^{\alpha}s^{\alpha-3}+M\lambda_2^{\alpha}s^{2\alpha-3})}{(s+\xi^{2})(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\xi^{2}+\xi^{2}\lambda_3^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{\alpha}s^{2\alpha})}\right) \qquad (3-16)$$

Now, taking the part $\frac{1}{(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+\xi^{2}+\xi^{2}\lambda_3^{\beta}s^{\beta}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{\alpha}s^{2\alpha})}$

$$=\frac{1}{\lambda_1^{\alpha}\left(\frac{s}{\lambda_1^{\alpha}}+s^{\alpha+1}+\frac{\lambda_2^{\alpha}s^{2\alpha+1}}{\lambda_1^{\alpha}}+\frac{\xi^{2}}{\lambda_1^{\alpha}}+\frac{\xi^{2}\lambda_3^{\beta}s^{\beta}}{\lambda_1^{\alpha}}+\frac{M}{\lambda_1^{\alpha}}+Ms^{\alpha}+\frac{M\lambda_2^{\alpha}s^{2\alpha}}{\lambda_1^{\alpha}}\right)}$$

And, by using $\left(\frac{1}{z+a}=\sum_{k=0}^{\infty}(-1)^{k}\,\frac{z^{k}}{a^{k+1}}\right)$ we get

$$\frac{1}{\lambda_1^{\alpha}\left(\frac{s}{\lambda_1^{\alpha}}+s^{\alpha+1}+\frac{\lambda_2^{\alpha}s^{2\alpha+1}}{\lambda_1^{\alpha}}+\frac{\xi^{2}}{\lambda_1^{\alpha}}+\frac{\xi^{2}\lambda_3^{\beta}s^{\beta}}{\lambda_1^{\alpha}}+\frac{M}{\lambda_1^{\alpha}}+Ms^{\alpha}+\frac{M\lambda_2^{\alpha}s^{2\alpha}}{\lambda_1^{\alpha}}\right)}$$

$$=\frac{1}{\lambda_1^{\alpha}}\sum_{m=0}^{\infty}(-1)^{m}\,\frac{\left(\frac{s}{\lambda_1^{\alpha}}+\frac{\lambda_2^{\alpha}s^{2\alpha+1}}{\lambda_1^{\alpha}}+\frac{\xi^{2}\lambda_3^{\beta}s^{\beta}}{\lambda_1^{\alpha}}+Ms^{\alpha}+\frac{M\lambda_2^{\alpha}s^{2\alpha}}{\lambda_1^{\alpha}}\right)^{m}}{\left(s^{\alpha+1}+\frac{1}{\lambda_1^{\alpha}}(\xi^{2}+M)\right)^{m+1}}$$

Now, taking the part $\left(\frac{s}{\lambda_1^{\alpha}}+\frac{\lambda_2^{\alpha}s^{2\alpha+1}}{\lambda_1^{\alpha}}+\frac{\xi^{2}\lambda_3^{\beta}s^{\beta}}{\lambda_1^{\alpha}}+Ms^{\alpha}+\frac{M\lambda_2^{\alpha}s^{2\alpha}}{\lambda_1^{\alpha}}\right)^{m}$

$$=\left(\frac{s}{\lambda_1^{\alpha}}\right)(1+\lambda_2^{\alpha}s^{2\alpha}+\xi^{2}\lambda_3^{\beta}s^{\beta-1}+M\lambda_1^{\alpha}s^{\alpha-1}+M\lambda_2^{\alpha}s^{2\alpha-1})^{m}$$

And, by using ($(1+b)^k = \sum_{m=0}^{k}\frac{k!b^m}{m!l!}$) we get

$$(\frac{s}{\lambda_1^\alpha})^m(1+\lambda_2^\alpha s^{2\alpha}+\xi^2\lambda_3^\beta s^{\beta-1}+M\lambda_1^\alpha s^{\alpha-1}+M\lambda_2^\alpha s^{2\alpha-1})^m$$

$$=\sum_{l=0}^{m}\frac{m!}{l!(m-l)!}(\lambda_2^\alpha s^{2\alpha}+\xi^2\lambda_3^\beta s^{\beta-1}+M\lambda_1^\alpha s^{\alpha-1}+M\lambda_2^\alpha s^{2\alpha-1})^l$$

$$=(\frac{s}{\lambda_1^\alpha})^m\sum_{l=0}^{m}\frac{m!}{l!(m-l)!}(\lambda_2^\alpha s^{2\alpha})^l(1+\frac{\xi^2\lambda_3^\beta s^{\beta-1-2\alpha}}{\lambda_2^\alpha}+\frac{M\lambda_1^\alpha s^{-\alpha-1}}{\lambda_2^\alpha}+Ms^{-1})^l$$

$$=(\frac{s}{\lambda_1^\alpha})^m\sum_{l=0}^{m}\frac{m!}{l!(m-l)!}(\lambda_2^\alpha s^{2\alpha})^l\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}(\frac{\xi^2\lambda_3^\beta s^{\beta-1-2\alpha}}{\lambda_2^\alpha}+\frac{M\lambda_1^\alpha s^{-\alpha-1}}{\lambda_2^\alpha}+Ms^{-1})^j$$

$$=(\frac{s}{\lambda_1^\alpha})^m\sum_{l=0}^{m}\frac{m!}{l!(m-l)!}(\lambda_2^\alpha s^{2\alpha})^l\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}(\frac{M}{s})^l(1+\frac{\xi^2\lambda_3^\beta s^{\beta-2\alpha}}{M\lambda_2^\alpha}+\frac{\lambda_1^\alpha s^{-\alpha}}{\lambda_2^\alpha})^j$$

$$=(\frac{s}{\lambda_1^\alpha})^m\sum_{l=0}^{m}\frac{m!}{l!(m-l)!}(\lambda_2^\alpha s^{2\alpha})^l\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}(\frac{M}{s})^l\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}(\frac{\xi^2\lambda_3^\beta s^{\beta-2\alpha}}{M\lambda_2^\alpha}+\frac{\lambda_1^\alpha s^{-\alpha-1}}{\lambda_2^\alpha})^i$$

$$=(\frac{s}{\lambda_1^\alpha})^m\sum_{l=0}^{m}\frac{m!}{l!(m-l)!}(\lambda_2^\alpha s^{2\alpha})^l\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}(\frac{M}{s})^l\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}(\frac{\lambda_1^\alpha s^{-\alpha}}{\lambda_2^\alpha})^i(1+\frac{\xi^2\lambda_3^\beta s^{\beta-2\alpha}}{M\lambda_1^\alpha})^i$$

$$=(\frac{s}{\lambda_1^\alpha})^m\sum_{l=0}^{m}\frac{m!}{l!(m-l)!}(\lambda_2^\alpha s^{2\alpha})^l\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}(\frac{M}{s})^l\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}(\frac{\lambda_1^\alpha s^{-\alpha}}{\lambda_2^\alpha})^i\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}(\frac{\xi^2\lambda_3^\beta s^{\beta-2\alpha}}{M\lambda_1^\alpha})^d$$

Hence, the Eq.(3-16) can be written under the form of a series as

$$\bar{U}_s(\xi,s)=\sqrt{\frac{2}{\pi}}\{[\frac{1}{\xi s^3}-\frac{1}{\xi^3 s^2}+(\frac{1}{s}-\frac{1}{s+\xi^2})\frac{1}{\xi^5}]-\xi(\lambda_1^{\alpha}s^{\alpha-2}+\lambda_2^{\alpha}s^{2\alpha-2}-\lambda_3^{\beta}s^{\beta-2}+Ms^{-3}$$

$$+M\lambda_1^{\alpha}s^{\alpha-3}+M\lambda_2^{\alpha}s^{2\alpha-3})\sum_{m=0}^{\infty}(-1)^m\frac{\sum_{l=0}^{m}\frac{1}{l!(m-l)!}\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}}{(s+\xi^2)(s^{\alpha+1}+\frac{1}{\lambda_1^{\alpha}}(\xi^2+M))^{m+1}}\}$$

$$\lambda_1^{\alpha(-m+i-d-1)}\ \lambda_2^{\alpha(l-i)}\ \lambda_3^{\beta d}\ M^{j-d}\ \xi^{2d}\ m!\,s^{\delta} \qquad (3-17)$$

In which $\delta=m+2\alpha l-j-\alpha i+\beta d-\alpha d.$

Now, applying the inversion formula term by term for the Laplace transform, Eq.(3-17) yields

$$U_s(\xi,\tau)=\sqrt{\frac{2}{\pi}}\,[\frac{\tau^2}{\xi}-\frac{\tau}{\xi^3}+\frac{1}{\xi^5}\,(1-\exp(-\xi^2\tau)]$$

$$-\int_0^{\tau}\xi\sqrt{\frac{2}{\pi}}\sum_{m=0}^{\infty}(-1)^m\sum_{l=0}^{m}\frac{1}{l!(m-l)!}\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}$$

$$\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}\,\lambda_1^{\alpha(-m+i-d-1)}\;\lambda_2^{\alpha(l-i)}\;\lambda_3^{\beta d}$$

$$M^{\,j-d}\;\xi^{2d}\times[\lambda_1^{\alpha}\sigma^{(\alpha+1)m+(3-\delta)-1}\;E_{(\alpha+1),(3-\delta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})$$

$$+\lambda_2^{\alpha}\;\sigma^{(\alpha+1)m+(3-\alpha-\delta)-1}\;E_{(\alpha+1),(3-\alpha-\delta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\;\sigma^{\alpha+1})$$

$$-\lambda_3^{\beta}\;\sigma^{(\alpha+1)m+(3+\alpha-\beta-\delta)-1}\;E_{(\alpha+1),(3+\alpha-\beta-\delta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})$$

$$+M\;\sigma^{(\alpha+1)m+(4-\delta+\alpha)-1}\;E_{(\alpha+1),(4-\delta+\alpha)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})$$

$$+M\,\lambda_1^{\alpha}\,\sigma^{(\alpha+1)m+(4-\delta)-1}\;E_{(\alpha+1),(4-\delta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})$$

$$+M\,\lambda_2^{\alpha}\,\sigma^{(\alpha+1)m+(4-\alpha-\delta)-1}\;E_{(\alpha+1),(4-\alpha-\delta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})]$$

$$*\exp(-\xi^2(\tau-\sigma)]\,d\sigma \qquad (3-18)$$

Where "$*$" represents the convolution of two functions and

$$E_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\lambda n+\mu)},\quad \lambda,\mu>0, \qquad (3-19)$$

Denotes the generalized Mittag-Leffler function with

$$E_{\lambda,\mu}^{(k)}(z)=\frac{d^k}{dz^k}\;E_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{(n+k)!z^n}{n!\Gamma(\lambda n+\lambda k+\mu)}. \qquad (3-20)$$

Here, we used the following property of the generalized Mittad-Leffler function[16]

$$L^{-1}\{\frac{k!s^{\lambda-\mu}}{(s^{\lambda}\mp c)^{k+1}}\}=t^{\lambda k+\mu-1}E_{\lambda,\mu}^{(k)}(\mp ct^{\lambda}),\quad (\mathrm{Re}(s)\succ|c|^{1/\lambda}). \qquad (3-21)$$

Finally, inverting (3-18) by the Fourier transform [18] we find for the velocity $U(\xi,\tau)$ the expression

$$U(\eta,\tau)=U_N(\eta,\tau)-\frac{2}{\pi}\int_0^{\infty}\int_0^{\tau}\frac{\sin(\xi\eta)}{\xi}\sum_{m=0}^{\infty}(-1)^m\sum_{l=0}^{m}\frac{1}{l!(m-l)!}$$

$$\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}\lambda^{\alpha(-m+i-d-1)}\ \lambda_2^{\alpha(l-i)}\ \lambda_3^{\beta d}\ M^{j-d}$$

$$\xi^{2d}\times[\lambda_1^{\alpha}\sigma^{(\alpha+1)m+(3-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(3-\delta))}$$

$$+\lambda_2^{\alpha}\sigma^{(\alpha+1)m+(3-\alpha-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(3-\alpha-\delta)}$$

$$+\lambda_3^{\beta}\sigma^{(\alpha+1)m+(3+\alpha-\beta-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(3+\alpha-\beta-\delta))}$$

$$+M\sigma^{(\alpha+1)m+(4-\delta+\alpha)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(4-\delta+\alpha))}$$

$$+M\lambda_1^{\alpha}\sigma^{(\alpha+1)m+(4-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(4-\delta))}$$

$$+M\lambda_2^{\alpha}\sigma^{(\alpha+1)m+(4-\alpha-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(4-\alpha-\delta))}]$$

$$*\exp(-\xi^2(\tau-\sigma)\,d\sigma\,d\xi \qquad (3-22)$$

Whence,

$$U_N(\eta,\tau)=\tau^2-\frac{4\tau}{\pi}\int_0^\infty \frac{\sin(\xi\eta)}{\xi^3}d\xi+\frac{4}{\pi}\int_0^\infty (1-\exp(-\xi^2\tau)\frac{\sin(\xi\eta)}{\xi^5}d\xi=32\tau^2 i^4 Erfc(\frac{\eta}{2\sqrt{\tau}}), \quad (3-23)$$

Represents the velocity field corresponding to a Newtonian fluid performing the same motion.

In the above relation $i^n Erfc(.)$ are the integrals of the complementary error function of Gauss.

(3-3)Results and discussion:

This section displays the graphical illustration velocity field for the flows analyzed in this investigation. We interpret these results with respect to the variation of emerging parameters of interest. The exact analytical solutions for accelerated flows have been obtained for a Burgers' fluid and a comparison is mad with the results for those of the fractional Oldroyd-B fluid.

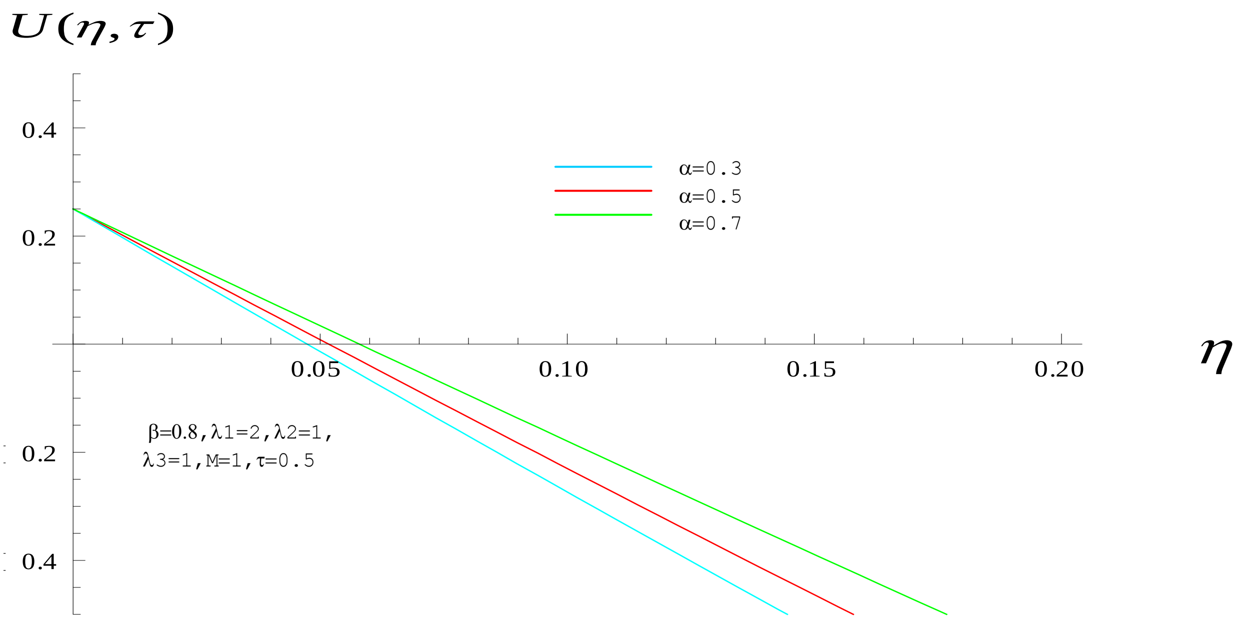
Fig. (3-1) is prepared to show the effects of non-integer fractional parameters α on the velocity field, as well as a comparison between the fractional Oldroyd-B fluid and fractional Burgers' fluid for fixed values of other parameters. . It is observed that for time $\tau = 0.5$ the velocity will increase by the increase in the parameter α. Moreover, for time $\tau = 0.5$ the velocity profiles for an Oldroyd-B fluid are greater than those for a Burgers' fluid. Its also observed that for time $\tau = 0.5$ the velocity profiles for Burgers' fluids approach the velocity profile of the fractional Oldroyd-B fluid and after some time it will become the same. Thus, it's obvious that the relaxation and retardation times and the orders of the fractional parameters have strong effects on the velocity field.

Fig. (3-2) is prepared to show the effects of non-integer fractional parameters β on the velocity field, as well as a comparison between the fractional Oldroyd-B fluid and fractional Burgers' fluid for fixed values of other parameters. It is observed that for time $\tau = 0.5$ the velocity will increase by the increase in the parameter β. Moreover, for time $\tau = 0.5$ the velocity profiles for an Oldroyd-B fluid are greater than those for a Burgers' fluid.

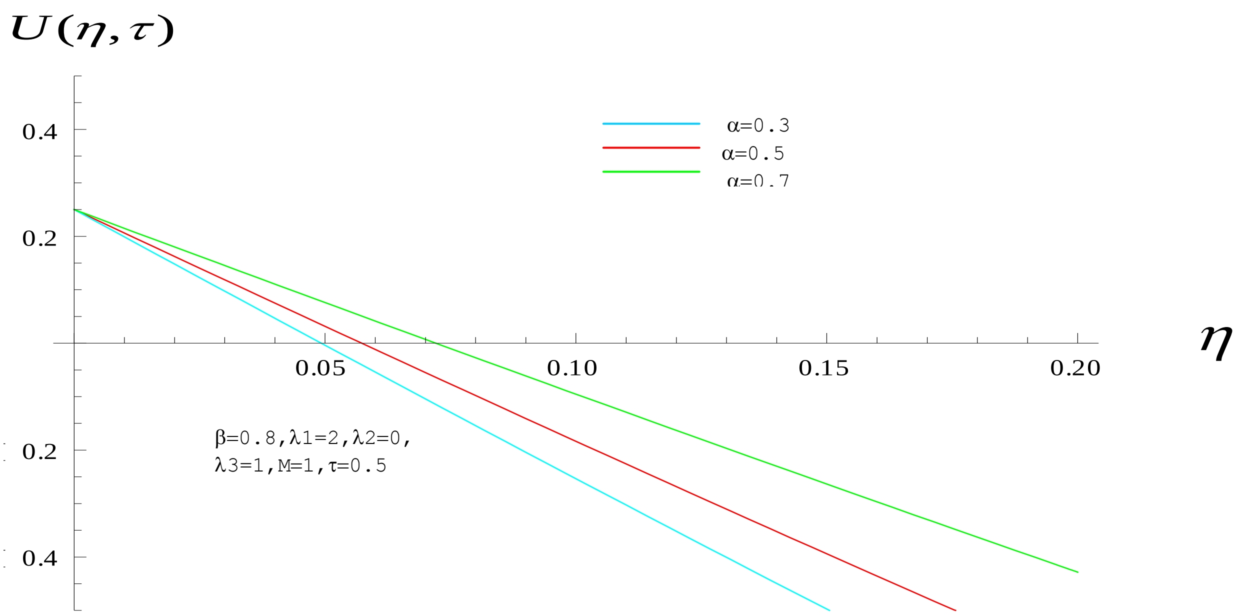
Fig. (3-3) shows the effects of new material parameter on the velocity field for fixed values of other parameters. It is observed that for time $\tau = 1$ the velocity will decrease by the increase in new material parameter λ_2.

Fig. (3-4) shows the variation of time τ on the velocity field for fixed values of other parameters. It's observed that the velocity will increase by the increase in time and after some time it will become the same.

Fig. (3-5) shows the velocity changes with the fractional parameters and the magnetic field parameter. It is observed that for $\alpha \leq 0.2$ the velocity will decrease by the increase in the magnetic field M. However, one can see that an increase in the magnetic field M for $\alpha \leq 0.6$ has same effect to that of $\alpha \leq 0.2$.

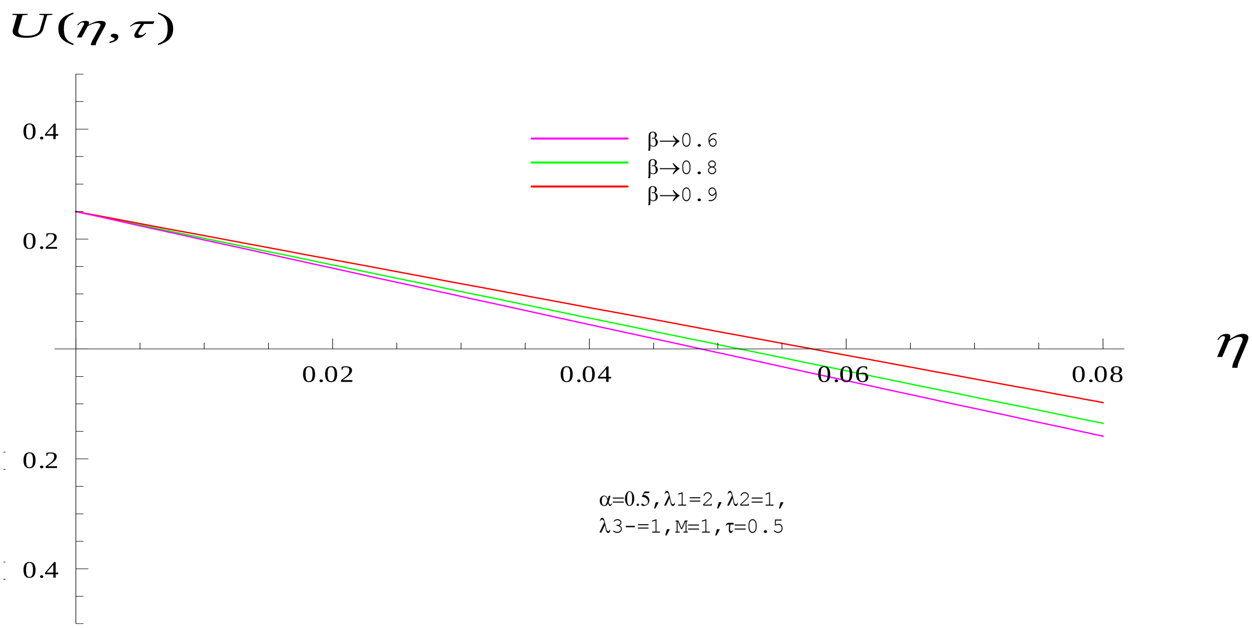


a) Burgers' model

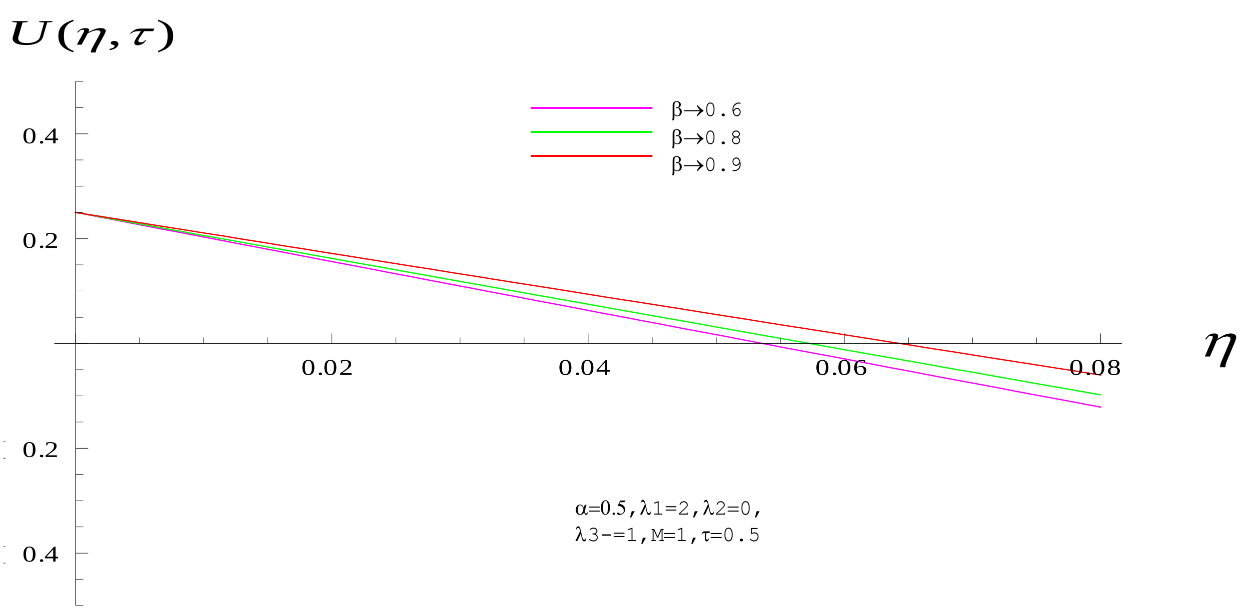


b) Oldroyd-B fluid

Fig.(3-1): Velocity $U(\eta,\tau)$ versus η for different values of α when other parameters are fixed.



a)Burgers' model



b)Oldroyd-B fluid

Fig.(3-2): Velocity $U(\eta,\tau)$ versus η for different values of β when other parameters are fixed.

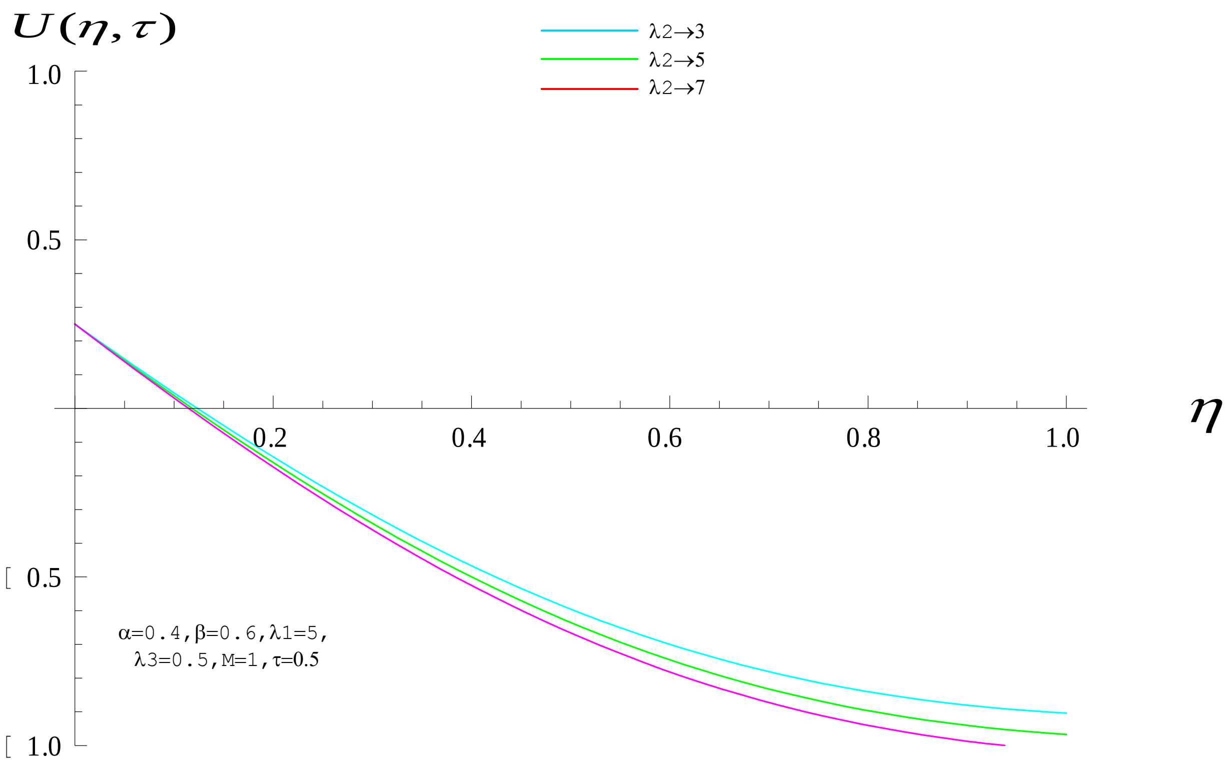


Fig.(3-3): Velocity $U(\eta,\tau)$ versus η for different values of λ_2 when other parameters are fixed.

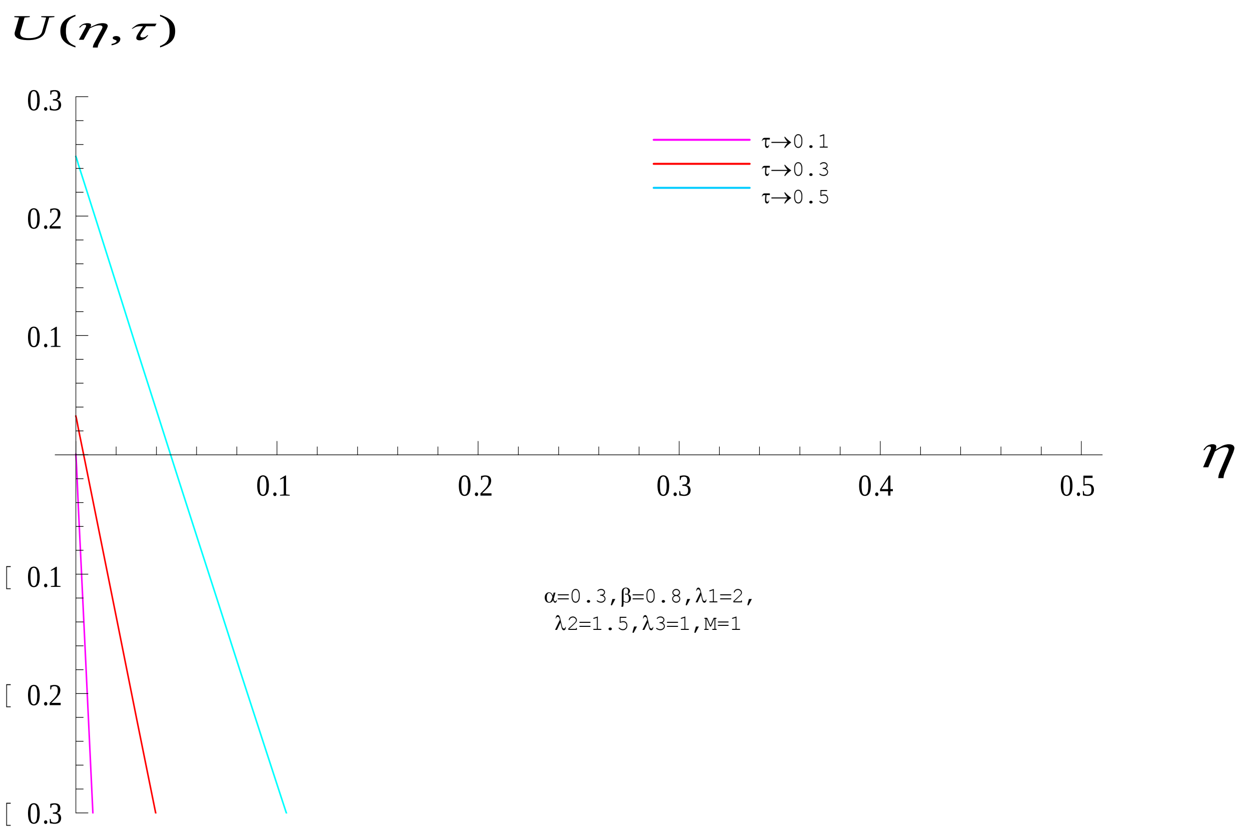


Fig.(3-4): Velocity $U(\eta,\tau)$ versus η for different values of τ when other parameters are fixed.

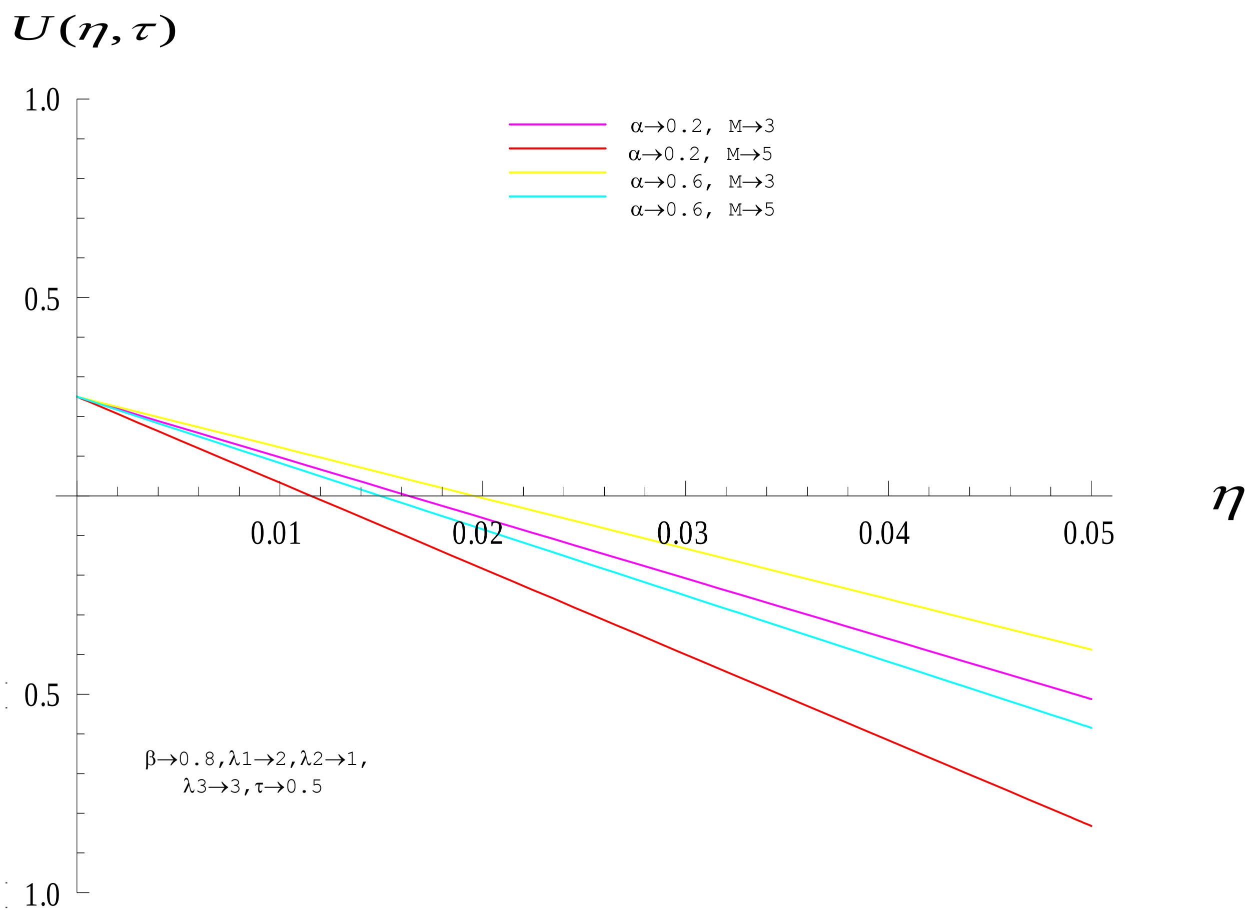


Fig.(3-5): Velocity $U(\eta,\tau)$ versus η for different values of α, M when other parameters are fixed.

Further Work

In what follow we give some suggestions for further work:

1- We solve the problem in two dimensions; one can resolve it in three dimensions.

2- We study the effect of MHD on the velocity field; one can study the effect of MHD on the Shear stress and Shear strain.

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